# ELEMENTARY CALCULUS
## AND
# COORDINATE GEOMETRY

BY

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PART I



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# PREFACE

This book is based on the belief that it is desirable to round off the ordinary school mathematics course with an introduction to the chief methods of the Calculus. To be practicable such an introduction must not assume any great knowledge of Algebra and must proceed as directly as possible to the two main problems, the problem of the tangent to a curve or the rate of change of one variable with another in a formula, and the problem of calculating areas, volumes, etc., when part of the boundary is curved. Part I of the book is intended to be a course of this kind; Part II is a continuation, intended mainly as a first course for specialists in Science or Mathematics, in which the concepts of Part I are gradually refined and the methods are extended.

It was not easy to decide how deeply to probe the difficulties underlying the two main problems in Part I. To emphasize these at too early a stage may produce inhibitions; to omit any discussion misses an opportunity. My solution is based on the axiom that it is easier for the teacher to omit parts of the bookwork than to supply some that is not there. Particularly in the chapter on summation the arguments have been presented so that the simpler parts only may be taken at a first reading if this is preferred.

Since Part I is concerned more with ideas than with methods, new notation is introduced sparingly and, as far as possible, not at the same time as the ideas. The notation $\int_a^b dx$ has such obvious advantages that any attempt to do without it is likely to prove a gratuitous handicap.

I have convinced myself that beginners take to integration more readily if the differential $dx$ has been defined previously in its own right. This dictates the notation $dy/dx$ for the derivative, and other notations are not used at all in Part I.

As far as possible each short piece of bookwork has its own set of simple and direct exercises. At the end of each chapter there are Miscellaneous Exercises arranged in three categories X, Y, and Z. The X Exercises are arranged in sets of ten, each set designed to cover the work of the previous chapter. These exercises are short and simple and are intended for consolidation. As soon as convenient the pupils should proceed to the Y Exercises which demand rather more sustained effort but are

mainly numerical and do not require any great algebraic skill The Z Exercises are a mixture of drill exercises, harder exercises involving more algebra, and some exercises of general interest which may be omitted where time is short

The algebra throughout both parts of the book is real, and this fact is particularly apparent in the sections on locating the roots of equations

In Part II technique assumes greater importance and the notation of Part I is supplemented with others The inductive method is still used in breaking new ground, but proofs of some of the results reached inductively in Part I are also given An effort has been made to match the standard of rigour and precision of statement with the growth of the pupil's mental powers I hope that this part of the book will give the pupil a sense of increased power and fuller understanding and will provide a smooth transition to the more advanced text books

In Part I Coordinate Geometry and Calculus march side by side, each illustrating the processes of the other and only the parts of Coordinate Geometry which are useful in this respect are included there In Part II it seemed more useful to develop the two subjects systematically in separate chapters The main purpose of the work on Coordinate Geometry is to introduce the general principles of the subject, but its applications to the Parabola, Ellipse, and Hyperbola are included, chiefly in the form of exercises, partly for interest, and partly because they may be useful for scientists A disadvantage of presenting bookwork as a series of exercises is that reference to it becomes less easy For this reason the results proved in the exercises are included in the index

Chapter X contains an account of the approximate solution of equations which is unusually detailed for a book of this kind For this idea I am greatly indebted to Professor T A A Broadbent who not only made the suggestion but outlined the method of approach I hope that other teachers will find the suggestion as valuable as I have done In addition to providing an introduction to the first principles of computation, it also satisfies a felt need, the question 'What do we do if the degree of the equation is greater than two?' is, in my experience often asked at about this stage The teacher has the further satisfaction of knowing that he is effectively preparing the ground for the thorough investigation of the limit of a sequence

I have also to thank Professor Broadbent for reading the whole manuscript in its original form and making many valuable suggestions In most of the exercises, the arithmetic has been

kept to a minimum, but some are based on 'real' data, and these cannot be expected to 'come out'. I acknowledge gratefully the permission to reproduce the facts upon which these exercises are based. I am also conscious that the exercises owe much to other authors and to the examiners of the various School and Higher Certificate Examinations. The kind permission of the Southern Railway to include on page 6 a drawing of part of their blueprint of an actual run will, I hope, interest the younger readers.

Finally, I wish to thank the staff of the Clarendon Press for their care in the production of the book.

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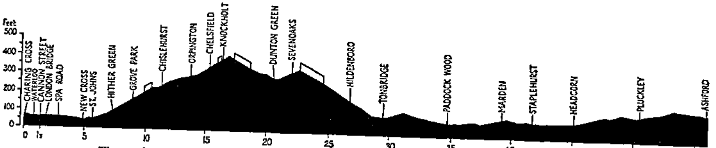



The gradients of the main line of the British Railways (Southern Region) from London to Ashford


Velocity distance graph of the 4 34 p m passenger express from Cannon Street to Ashford on 2 June 1916

# CONTENTS

# LIST OF SYMBOLS

$=$, equals
$\equiv$, is the same as, or identically equal to
$\doteq$, approximately equal to
$<$, less than
$>$, greater than
$\leqslant$, less than or equal to
$\geqslant$, greater than or equal to
$\not>$, not greater than ($\equiv \leqslant$)
$\not<$, not less than ($\equiv \geqslant$)
$\sqrt{x}$, the positive square root of $x$
$^\circ$ degrees
$^c$ radians
* may be omitted at a first reading

For convenience in printing, fractions like $\dfrac{2x+1}{3}$ will sometimes be written in the form $(2x+1)/3$ The influence of the oblique stroke (*solidus*) is felt only by the numbers, products, or brackets immediately to the left and right of it, and does not extend across positive or negative signs connecting different terms of the expression So,

$$x^2+2x-1/2x^2 = x^2+2x-\frac{1}{2x^2},$$

$$(x^2+2x+1)/x^2 = \frac{x^2+2x+1}{x^2},$$

$$1/x^2-2+3x^2 = \frac{1}{x^2}-2+3x^2$$

# I

# CONSTANT GRADIENTS

## 1.1. The graph of an equation of the first degree

An equation such as $3y-2x = 1$, which contains no terms of degree higher than 1 (i.e. no terms like $x^2$, $xy$, $y^2$, etc.) is called an *equation of the first degree*. To draw the graph of such an equation we make a table of values by giving $x$ any values we like and calculating the corresponding values of $y$ which make the equation true. Thus for the equation $3y-2x = 1$ we might make the following table of values:

| | | | | |
|---|---|---|---|---|
| $x$ | $-2$ | $0$ | $1$ | $3$ |
| $y$ | $-1$ | $\frac{1}{3}$ | $1$ | $2\frac{1}{3}$ |

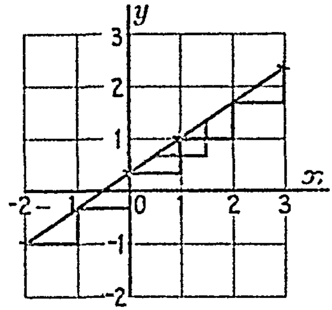


Fig. 1.1

We now plot the corresponding values of $x$ and $y$ and join the points so obtained by a smooth line. In the present case we can see that this line is straight and it may be drawn with a ruler.

You probably know that the graph of an equation of the first degree is always a straight line. For this reason the equation is also called a *linear equation*. The fact that a linear equation always has a straight line graph is most useful. For example, it means that the graph may be drawn with a ruler from a table of values which contains only *two* pairs of values.

To show that the graph of $3y-2x = 1$ is a straight line we write the equation in the form $y = (2x+1)/3$.

Let $x$ increase by 1 and let $y$ change to $Y$. Then

$$Y = \frac{2(x+1)+1}{3} = \frac{2x+2+1}{3} = \frac{2x+1}{3}+\frac{2}{3}.$$

Therefore, whenever $x$ increases by 1, $y$ increases by $\frac{2}{3}$. Note that this increase of $y$ does not depend upon $x$. Whatever value of $x$ we start from, an increase of 1 in $x$ results in an increase of $\frac{2}{3}$ in $y$. This shows that the graph is straight. The right-angled triangles drawn on Fig. 1.1 are congruent and their hypotenuses all have the same slope.

The number $\frac{2}{3}$ is called the *gradient* of the line $3y-2x = 1$. It is clear that it measures the steepness of the line. We might say that the line rises $\frac{2}{3}$ in 1, or 2 in 3. [Verify that an increase of 3 in $x$ means an increase of 2 in $y$.]

The gradient is easily found from the table of values From the first two entries we see that when $x$ increases by 2 (from $-2$ to 0) $y$ increases by $\frac{4}{3}$ (from $-1$ to $\frac{1}{3}$) Thus the fraction

$$\frac{\text{increase of } y}{\text{increase of } x} = \frac{4/3}{2} = \frac{2}{3}$$

[Verify that $\frac{2}{3}$ is the gradient obtained by using any other two entries in the table ]

More generally, let any two entries in the table of values be *denoted by the following letters*

$$\begin{array}{ccc} x & x_1 & x_2 \\ y & y_1 & y_2 \end{array}$$

Then, since $x_1$, $y_1$ and $x_2$ $y_2$ satisfy the equation $3y = 2x+1$

$$3y_1 = 2x_1+1$$

and $$3y_2 = 2x_2+1$$

Hence $$3(y_2-y_1) = 2(x_2-x_1)$$

or $$\frac{y_2-y_1}{x_2-x_1} = \frac{2}{3}$$

Thus the difference of any two values of $x$ divided into the difference of the two corresponding values of $y$ gives the gradient of the line

EXAMPLE If the following pairs of values are plotted find the gradient of the straight line joining them

| | | |
|---|---|---|
| $x$ | 3 | 8 |
| $y$ | 4 | 15 5 |

*Solution* The required gradient is

$$\frac{15\ 5-4}{8-3} = \frac{11\ 5}{5} = 2\ 3$$

The order in which the pairs of values are taken is immaterial Thus the gradient is also given by

$$\frac{4-15\ 5}{3-8} = \frac{-11\ 5}{-5} = 2\ 3$$

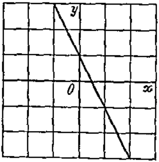



FIG 1 2

Now consider the graph of

$$y = 1-2x$$

[Draw it on graph paper ] Let $x$ increase by 1 Using the same notation as before, $y$ changes to

$$Y = 1-2(x+1) = 1-2x-2$$

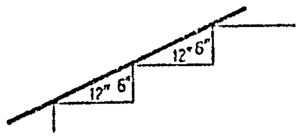
Thus an increase of 1 in $x$ means a decrease of 2 in $y$

We say that the gradient of this line is $-2$. If $x_1, y_1$ and $x_2, y_2$ are two corresponding pairs of values,

$$\begin{aligned} y_1 &= 1-2x_1 \\ y_2 &= 1-2x_2 \\ y_2-y_1 &= -2(x_2-x_1) \\ \frac{y_2-y_1}{x_2-x_1} &= -2. \end{aligned}$$

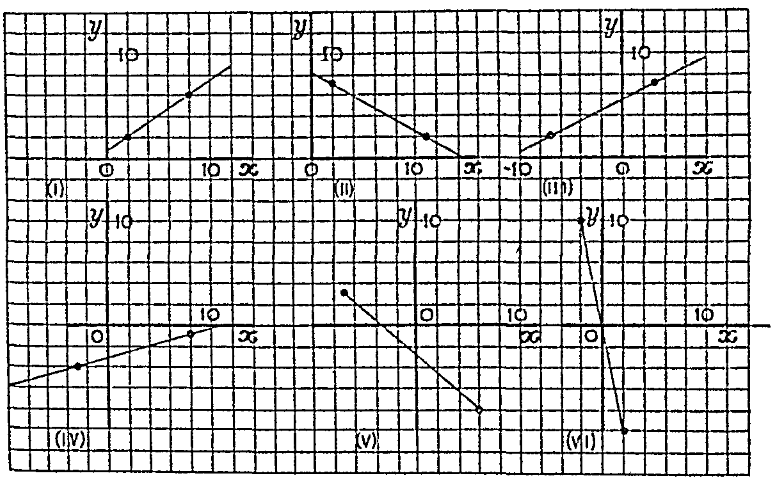
EXERCISE 1.A

1. On a flight of steps the 'tread' is 12 in. and the 'rise' is 6 in. What would be the gradient of a plank laid straight up and down the steps?




2–7. Find the gradient of each of the following lines. Draw a graph of each.

2. $y = \frac{1}{2}x+3$.
3. $2y = 4-x$.
4. $2y = 4x-1$.
5. $3y+5x = 0$.
6. $5y+10x = 1$.
7. $6x-3y+4 = 0$.




For clearness alternate lines of the graph paper are omitted. Points not at an intersection are to be taken as exactly midway between the nearest ruled lines.

8. Find the gradients of the lines marked (i) to (vi) in the figure above.

9–16 Plot the following pairs of values and find the gradient of the line joining them

9 $x = 1, y = 2, x = 3 \ y = 4.$

10 $x = 2, y = 2\frac{1}{2}, x = 3, y = 4\frac{1}{2}$

11 $x = -1, y = 1, x = 1, y = 4$

12 $x = -1, y = -2, x = 4, y = 3$

13 $x = -3 \ y = 2, x = 0, y = 1$

14 $x = -2, y = 3, x = 2, y = 0$

15 $x = 2, y = 4, x = -2, y = 1$

16 $x = 1, y = \frac{1}{2}, x = 3, y = \frac{1}{2}$

17 A man climbing a hill pauses to admire the view at horizontal intervals of 200 ft His horizontal distance from the starting point and his height above his starting point at each pause are given in the following table

| Horizontal distance (ft ) | 200 | 400 | 600 | 800 | 1000 | 1200 | 1400 |
|---|---|---|---|---|---|---|---|
| Height (ft ) | 70 | 160 | 300 | 460 | 560 | 615 | 660 |

How far has he travelled horizontally when he comes to the steepest part of the hill ?

18 A ship takes soundings in fathoms with the result shown in the table With these data is it reasonable to believe that the section of the sea bottom over which the ship passes is a straight line ?

| Depth (fathoms) | 42 | $37\frac{1}{2}$ | 30 | 27 | 21 |
|---|---|---|---|---|---|
| Distance travelled (cables) | 0 | 6 | 16 | 20 | 28 |

## 1 2 The direction angle of a line

Draw the graph of $3y-4x+2 = 0$ and verify that the gradient is $\frac{4}{3}$

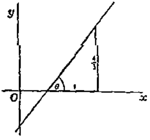


Fig 1 3

If $\theta$ is the angle from the positive $x$ axis to the line (measured counter clockwise according to the usual con vention of trigonometry) it is clear from Fig 1 3 that $\tan\theta = \frac{4}{3}$ From the tables $\theta = 53° \ 08'$

We shall call this angle the *direction angle* of the line

Now consider the graph of

$$2y+4x+5 = 0$$

Verify that its gradient is $-2$

If $\theta$ is the angle measured counter-clockwise from the positive $x$-axis to the line (i.e. the direction angle of the line)

$$\tan \theta = -2$$

and $$\theta = 116° \, 34'.$$

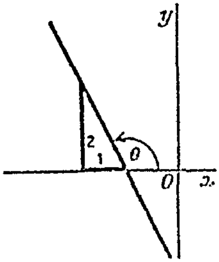



FIG. 1.4

Thus, in all cases, if the line has a gradient $g$, and its direction angle is $\theta$,

$$g = \tan \theta.$$

The term 'direction angle' is not in general use. A common description of this angle is 'the angle made by the line with the positive $x$-axis'. Note that through any point in the plane only one line can be drawn making a given angle with the *positive*

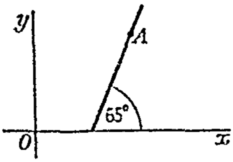



FIG. 1.5
Line through A making an angle of 65° with the positive $x$-axis (direction angle, 65°).

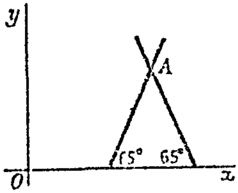
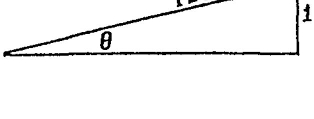



FIG. 1.6
Lines through A making an angle of 65° with the $x$-axis (direction angles, 65° and 115°).

$x$-axis but two lines can be drawn making a given angle with the $x$-axis.

## 1.3. The graph gradient compared with the gradients used in practice

The frontispiece shows the profile of the British Railways (Southern Region) main line from Cannon Street Station, London, to Ashford. This track, like all other railway tracks, is laid in sections, each of which has a constant gradient. For example, Grove Park station is on a section, just over 2 miles long, which has a gradient of 1 in 120. Now this gradient is not measured in the way already described for the gradient of a straight-line graph. The railway engineer is interested in distance measured along the track and for him a rising gradient of 1 in 120 means a vertical rise of 1 ft. in 120 ft. travelled along the track. The slope of the section passing through Grove Park station is therefore given by $\sin \theta = \frac{1}{120}$.



However, when a diagram like the frontispiece is made it is more convenient to measure track distances along a horizontal straight line as we do in graphs (You can see the scale of distance set out on a horizontal line ) To draw the sloping lines it is then convenient to know the gradient in the form $\tan\theta = k$, where $k$ is a suitable number In other words, it is convenient to change over to the graph gradient when the diagram is made Fortunately, in the cases which occur in practice, this is easy to do If you compare the sine and tangent tables you will find that the sines and tangents of angles from 0° to 2° 36′ are equal as far as 4 decimal figures The common value given for the sine and tangent of 2° 36′ is 0 0454 and this is a gradient of 454 in 10,000 or about 1 in 22 This is far steeper than any railway gradient Thus, for all practical purposes, the railway gradient may be used as if it were a graph gradient

If you are familiar with map reading you will know that the gradient of a hill is determined from a map by dividing the vertical rise between two contours (the Vertical Interval) by the distance on the ground equivalent to the *horizontal* distance between the contours (the Horizontal Equivalent) This gradient, $G$, is therefore connected with the slope by the equation $G = \tan\theta$ The 'map' gradient is therefore the same as the graph gradient

### Exercise 1 B

1–7 Find as accurately as tables permit the direction angles of the following lines

1 $y = 5x+1$ 2 $5y+x+10 = 0$
3 $3y+9x = 1$ 4 $2y = x$
5 $y+x = 0$ 6 $y = 5$
7 $x+1 = 0$

8–9 Find as accurately as tables permit the acute angle between the lines

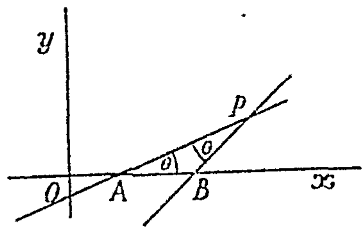
8 $3y-x = 2$, $2y-5x+3 = 0$
9 $2y+x = 1$, $2y-3x = 4$

10 At Orpington station, on the S R main line, whose approximate profile is shown in the frontispiece, the gradient of the track is 1 in 310 Find the angle between the track and the horizontal as accurately as 4 figure tables permit

11 Hildenborough station is on a section of the track which is 4 miles long and has a constant gradient of 1 in 122 What height, to the nearest foot, is lost by a train travelling from London to Ashford while it is on this section ?

12. What are the gradients of the two lines through the origin which make angles of 50° with the $x$-axis ? (Answers to the nearest $\frac{1}{10}$.)

13. The equation of $AP$ is $5y = 2x-6$. What is the gradient of $PB$ to the nearest $\frac{1}{100}$ ?




14. Find (a) the gradient, (b) the direction angle of the straight line joining (0, 4) and (5, 0).

15. Find the gradient and direction angle of each of the lines

$$y = x+1,$$
$$3y = 3x+10.$$

16. Find the acute angles between the following pairs of lines:

(a) $4y+5x = 0, y = 1$;

(b) $4y+5x = 0, x+1 = 0$;

(c) $4y+5x = 0, 4y-5x = 0$.

## 1.4. Families of lines

Consider the equation $y = mx+5$. If we give $m$ any definite value, say 2, we obtain $y = 2x+5$. The graph of this equation is a straight line. Draw this line, and on the same figure draw the lines obtained by giving $m$ the values $\frac{1}{2}$, $-1$, 0, and any other value you choose. Your figure shows that these lines stand in a certain geometrical relation to one another. What is it ? Because the lines are related in this way they are said to form a *family* and $y = mx+5$ is called the *equation of the family*. The equation of any member of the family is found by giving $m$ a suitable value. Is it possible to count the total number of lines in the family ?

Inspection of your figure will show that each line has its own direction. So the effect of changing $m$ in the equation of the family is to change the direction of the line chosen. It seems clear that the value of $m$ defines the direction of each line. We can confirm this by finding the gradient of $y = mx+5$. If $x$ increases by 1, $y$ increases by $m$. Therefore the gradient of $y = mx+5$ is $m$.

Now consider another family of lines which has the equation $y = 2x+k$, where the members of the family are found by giving numerical values to $k$. Draw the lines for $k = 1, 4, -2, 0, -5$. What is the geometrical relation between the lines ? What is the geometrical effect of changing the value of $k$ ?

### Exercise 1 C

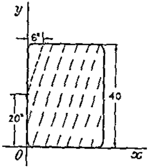
1 The equation of a family of lines is $y = mx-2$ Write down the equations of the four lines for which $m = \frac{1}{2}$, $m = 3$, $m = -4$, $m = -1$ Which of these lines is most nearly perpendicular to (*a*) the $x$ axis, (*b*) the $y$ axis?

2 Find the gradient of $y-3x+2 = 0$ Write down the gradients of the following lines (*a*) $y-3x = 1$, (*b*) $2y-6x = 5$, (*c*) $y = 3x-4$, (*d*) $9x-3y = 7$ What can you say about all these lines? Write down the equation of the family to which they belong

3 Write the line $3y-2x = 1$ in the form $y = \frac{2}{3}x+\frac{1}{3}$ and read off the gradient Check by giving $x$ an increase of 3

4–13 Write down the gradients of the following lines Check by drawing sketches (Definition A *sketch* is a quick figure drawn freehand to assist the understanding I find it quickest to sketch on graph paper)

| | | |
|---|---|---|
| 4 $3y-4x = 12$ | 5 $2y+6x = 8$ | 6 $3y = 6x-10$ |
| 7 $x+2y-4 = 0$ | 8 $1+x = 3y$ | 9 $2x+4y+5 = 0$ |
| 10 $y+5 = 0$ | 11 $y+x = 1$ | 12 $2y-3x+6 = 0$ |
| 13 $x/2+y/3 = 1$ | | |




14 The figure shows the paths of raindrops on a railway carriage window Find the gradient of the paths referred to the axes shown

15 Sketch the straight lines $y+2 = 0$, $x = 3$ What is the direction angle of each line?

16 Describe the families (*a*) $y-x = k$, (*b*) $y = k$, (*c*) $x = k$

17 Write down the equation of the line of the family $y = mx-2$ which is (*a*) perpendicular to $y = x-2$, (*b*) parallel to the $x$ axis

18 Write down the equations of the families of parallel lines whose gradients are (*a*) 4, (*b*) $-2$, (*c*) 0, (*d*) $\frac{1}{3}$

**1 5.** We have seen that two lines with equal gradients are parallel For example, all the lines of the family $y = 5x+k$ are parallel The direction angles of all these lines are also equal.

Now consider two lines whose gradients are equal but opposite in sign, e g $2y+x = 2$ and $2y-x = 1$ whose gradients are $-\frac{1}{2}$ and $+\frac{1}{2}$

If $\theta_1$ and $\theta_2$ are the direction angles of these lines

$$\tan\theta_1 = -\tfrac{1}{2} \quad \text{and} \quad \tan\theta_2 = \tfrac{1}{2}$$

Hence $\theta_1$ and $\theta_2$ are supplementary angles

$$[\theta_1 = 153^\circ\ 26',\ \theta_2 = 26^\circ\ 34]$$

The lines make equal acute angles with the $x$-axis but they are not parallel. Since the $y$-axis is perpendicular to the $x$-axis, the lines are also equally inclined to the $y$-axis. Hence two lines whose gradients are equal but opposite in sign are described as *equally inclined to the axes.* When a billiard ball strikes the cushion of the table, provided no spin has been given to it, the angle at which it rebounds is very nearly equal to the angle at which it approaches the cushion. If the cushion is taken as one of the axes of coordinates the two straight lines of which the path of the ball consists may be taken as equally inclined to the axes.

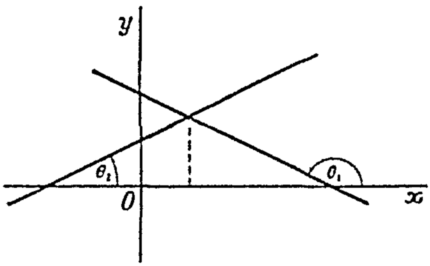


FIG. 1.7

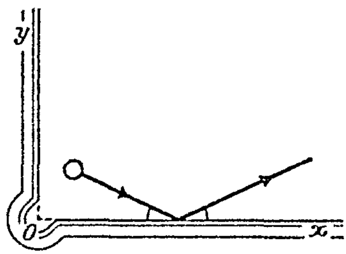


FIG. 1.8

If a stick is laid against a mirror, its reflection appears to be equally inclined to the mirror. If the stick does not touch the mirror its prolongation would meet the prolongation of the reflection at a point of the mirror. In geometry we call the line $A'B'$ the *reflection* of $AB$ in the line $OP$ if $AB$, $A'B'$ are *equally inclined* to $OP$ and *meet on* $OP$. The point $A'$ is called the *reflection* of $A$ if $OP$ is the perpendicular bisector of $AA'$.

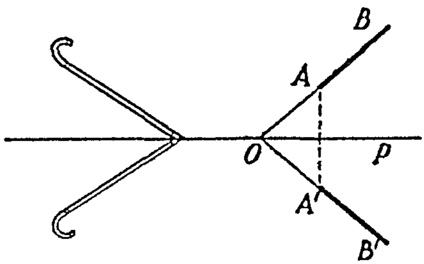


FIG. 1.9

## EXERCISE 1.D

1. What are the gradients of $3x+4y = 1$, $3x-4y = 1$? Describe these lines with respect to the axes.

2. What geometrical fact can be stated about the following pairs of lines?

(a) $y-2x+1 = 0$, $\quad y-2x-1 = 0$;

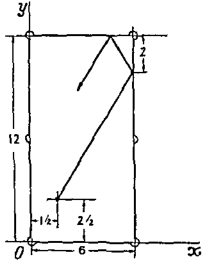
(b) $y-2x-1 = 0$, $\quad y+2x-1 = 0$;

(c) $5y+7x+3 = 0$, $\quad 7x+1 = 5y$.

3 What is the gradient of the reflection in the $x$ axis of the line (a) $y = 3x$, (b) $y + 2x = 0$?

4 What is the gradient of the reflection in the $y$ axis of $x+2y = 3$?

5 The figure shows a billiard table (dimensions in feet) A ball in the position shown is struck and follows the path marked by the arrow Assuming that each time the ball hits a cushion it rebounds at an equal angle, find the gradients of the three portions of the path and show that the first and third portions are parallel




6 Two of the following lines are parallel Which are they? (i) $2x+5y = 1$, (ii) $2x+4y = 3$, (iii) $4x+10y = 7$

7 Two of the following lines are equally inclined to the axes Which are they? (i) $2y-3x = 5$, (ii) $2y+3x = 1$, (iii) $y+3x = 7$

## 1 6 Coordinates

You may have noticed that the work of the preceding sections has become more and more geometrical We usually think of graphs as an aid to algebra, but it is now becoming clear that equations and gradients may be useful in discussing the geometry of straight lines This idea first occurred to the leading mathematicians of the world towards the end of the sixteenth century, and the credit for first developing it is usually given to the French mathematician Rene Descartes (1596–1650)

Suppose we wish to solve a problem in plane geometry involving points and straight lines, as for example, to prove that the medians of a triangle meet at a point In the plane of the figure we draw two perpendicular lines to serve as *coordinate axes*, one of these we call the $x$ axis and the other the $y$ axis, their point of intersection we call the *origin* On each axis we choose a scale for the measurement of distance Then every point in the plane has its position completely determined by two directed numbers giving the steps parallel to the axes by which the point may be reached from the origin the usual sign convention of graphs being observed These numbers are called the *coordinates* of the point

If $ABC$ is the triangle and the axes are drawn as shown in Fig. 1.10, $A$ may be reached from the origin, $O$, by a step of $-1$ along the $x$-axis followed by a step of $+3$ parallel to the $y$-axis (or by a step $+3$ along the $y$-axis followed by a step $-1$ parallel to the $x$-axis). The $x$-coordinate of $A$ is $-1$ and its $y$-coordinate is 3. We describe the point $A$ as the point $(-1, 3)$, always naming the $x$-coordinate before the $y$-coordinate. Verify that $B \equiv (-2\frac{1}{2}, -1\frac{1}{2})$ and $C \equiv (3, -1)$.

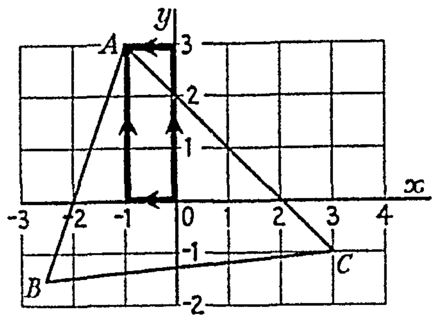


FIG. 1.10

The straight lines of the figure are the graphs of linear equations. But when we do geometry we may not be given these equations first. The line $AC$, for example, is defined as the join of the points $(-1, 3)$ and $(3, -1)$. We must then determine the equation from which $AC$ may be plotted. When the emphasis is upon geometry, this equation is called *the equation of the line AC*, just as when the emphasis is upon algebra $AC$ is called the graph of the equation.

There is a certain amount of freedom in choosing the axes. We can begin by laying down the $x$-axis. Then we can select any point of this line as origin. The $y$-axis is then determined, as it must be at right angles to the $x$-axis.†

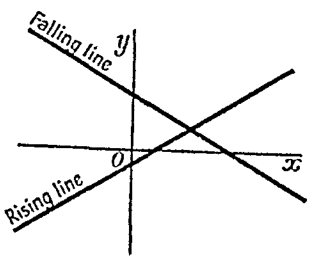


FIG. 1.11

The following convention will be used in drawing axes throughout this book.

The $x$-axis is drawn across the paper, with its positive sense towards the right.

The positive sense of the $y$-axis will be towards the top of the paper.

This does not limit our choice of axes and it does enable us to use certain descriptive phrases without ambiguity.

For example, as the eye travels from left to right across the page, following a line of positive gradient, it rises.

A line of positive gradient is called a *rising* line.

„ negative „ „ *falling* „

† Non-perpendicular (oblique) axes can be used, but the algebra is more complicated. In this book we shall always use perpendicular (rectangular) axes.

Again, this convention enables us to say that a line of gradient 2 is *steeper* than a line of gradient 1 Similarly a line of gradient −2 is steeper than a line of gradient −1

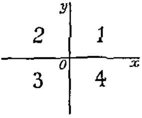



Fig 1 12

The coordinate axes divide their plane into four regions *called quadrants* These are numbered as shown in Fig 1 12 so that we may refer to them as first quadrant, second quadrant etc

The method of fixing position upon a surface by two coordinates is widely used, e g position at sea by latitude and longitude position on land by the coordinates of the National Grid (see Misc Ex 1 Z, No 2)

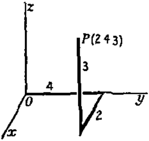



Fig 1 13

In space, *three* coordinates are necessary to fix the position of a point For example, the position of an electric lamp in a room may be defined by two coordinates determining the position on the ceiling from which it hangs, together with a third coordinate giving its depth below the ceiling In solid geometry we choose three coordinate axes, each one at right angles to the other *two and any point is defined by three* coordinates

## Exercise 1 E

1 Plot the points whose coordinates are (3, 1) (1, 3), (−1, 2), (3, −1) (3 4) (0, −1), (−3, 1), (3, −3) If your plotting is correct, all the points should lie on the sides of a triangle Rule in the triangle Which of the following points are inside the triangle? (1, 0), (−1, −1), (−2, 0), (0, 2), (0, 0) How many of the 8 points not on the axes are in (*a*) the first, (*b*) the second, (*c*) the third, (*d*) the fourth quadrant?

2 What can you say about the $x$ coordinates of points in the first and fourth quadrants?

3 In which quadrants do points have negative $y$ coordinates?

4 In which quadrants do points have $x$ and $y$ coordinates of the same sign?

5 Name the quadrant in which each of the following points is found (*a*) (−1, 2), (*b*) (1, 2), (*c*) (−1, −2), (*d*) (1, −2)

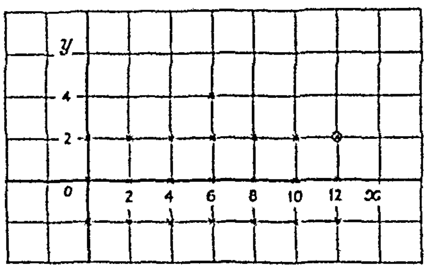
6. *Looping the loop.*† The position of an aircraft at various times with reference to a horizontal $x$-axis and a vertical $y$-axis (upward sense positive) is given in the following table:

| $t$ (sec.) | 0 | 0·5 | 1 | 1·5 | 2·0 | 2·5 | 3·0 | 3·5 | 4·0 | 5·0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ (ft.) | 218 | 140 | 70 | 6 | −40 | −70 | −82 | −85 | −79 | −52 |
| $y$ (ft.) | −126 | −140 | −132 | −107 | −68 | −25 | 15 | 57 | 95 | 136 |

| $t$ | 6·0 | 7·0 | 8·0 | 8·5 | 9·0 | 9·5 | 10·0 | 10·5 | 11·0 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | −12 | 38 | 91 | 120 | 138 | 141 | 127 | 97 | 50 |
| $y$ | 155 | 150 | 117 | 83 | 40 | −11 | −70 | −120 | −160 |

Plot the path of the aircraft and mark the times on your figure. At what part of the path is the speed least?

7. A platoon of 21 men and 1 sergeant is drawn up on the parade ground as shown in the sketch.




Axes are laid down, the unit on each axis being 1 pace. The distance between the ranks and between the men in the ranks is 2 paces. The sergeant is 2 paces in front of the centre of the front rank. Copy the sketch on graph paper.

Define, by coordinates, the position of (*a*) the right marker (⊗), (*b*) the sergeant.

What do you notice about (*c*) the $y$-coordinates of the men in any rank? (*d*) the $x$-coordinates of the three men in any file?

The sergeant gives the order 'Open order—March'. The sergeant and the front rank take 2 paces forward, the rear rank takes 2 paces backward. Define, by coordinates, the new position of (*e*) the right marker, (*f*) the centre man of the rear rank.

If the whole platoon moves 4 paces to the right, from its original position (in close order) every man's coordinates change by the same amount. (*g*) Find the change in each coordinate.

The platoon then marches forward 6 paces.

(*h*) Find the coordinates of the man who was originally at the origin.

(*i*) Also find the coordinates of the right marker.

*Miscellaneous Exercise 1.Z, Nos. 1, 2, 3 provide further practice with coordinates if required.*

† From Bairstow, *Applied Aerodynamics*, p. 233.

8 Find the steps parallel to the axes by which $B$ is reached from $A$ in the following cases

| | $A$ | $\rightarrow$ | $B$ |
|---|---|---|---|
| (a) | (3, 1) | | (8, 4) |
| (b) | (1, 2) | | (7, −2) |
| (c) | (−3, 4) | | (6, 1) |
| (d) | (−4, −3) | | (5, 5) |
| (e) | (4, 2) | | (−5, 3) |
| (f) | (3, 5) | | (0, −4) |
| (g) | (−2, −1) | | (10, −1) |
| (h) | (2, 2) | | (2, −9) |

9 A ship steers 10 miles NE followed by 8 miles SE Find the east step and north step by which the final position can be reached from the start (Answer correct to $\frac{1}{10}$ mile)

10 Find the gradients of the lines joining the following pairs of points State whether each is a rising or falling line

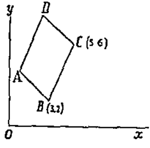
(a) (1, 2) and (11, 9), (b) (−3, −5) and (7, 8),
(c) (−2, 8) and (6, 4), (d) (−3, 5) and (2, −5),
(e) (−7, −2) and (11, −2), (f) (−4, −6) and (6, −5)

11 Write down the coordinates of the reflection of (3, 4) (a) in the $x$ axis, (b) in the $y$ axis

12 Find the gradients of the lines joining (1, 0) to (a) (0, 1), (b) (4, 6), (c) (−1, 8), (d) (0, −2) Which is the steepest line?

13 $A \equiv (2, 5)$ $B$ is the reflection of $A$ in the $y$ axis $C$ is the reflection of $B$ in the $x$ axis Find the gradients of the lines joining the origin, $O$, to (i) $A$, (ii) $C$ What can you infer from these results about $A$, $O$, $C$?

14 Prove that the points (1, 1), (5, 2), (7, $2\frac{1}{2}$) are in line




15 $ABCD$ is a parallelogram and $B \equiv (3, 2)$, $C \equiv (5, 6)$ What is the gradient of $AD$?

16 Show that the line joining (−1, −3) and (4, 6) is parallel to the line joining (2, 8) and ($3\frac{2}{3}$, 11)

17 Find the gradients and direction angles of the lines joining (0, 2) to (a) (2, 6 4), (b) (10, −1)

18 Find the angles of the triangle whose vertices are (0, 0), (0, 4), (3, 0)

19 Find the missing numbers in the following statements

(a) The line $3x-y+2=0$ passes through (1, —) and has gradient —

(b) The line $2y+5x-1=0$ passes through (2, —) and has gradient —

(*c*) The line $x = 4-3y$ passes through (—, 1) and has gradient—

(*d*) The line $y = 2$ cuts the $y$-axis where $y =$ — and has gradient —

(*e*) The line $3x+4y-1 = 0$ cuts the $x$-axis where $x =$ —

(*f*) The line $5x-y+15 = 0$ cuts the $y$-axis where $y =$ —

20. Find the points of intersection of $x/2+y/3 = 1$ with the coordinate axes.

21. Find the points of intersection of $3x+4y = 12$ with the coordinate axes.

22. Solve graphically the simultaneous equations $y = x+1$, $y+2x = 7$.

Define, by coordinates, the intersection of the straight lines $y = x+1$, $y+2x = 7$.

23. Find the coordinates of the intersection of the lines $y+x = 2$, $2y-x = 1$. Check by drawing the graphs.

24. Find the coordinates of the vertices of the triangle whose sides are $y+x = 0$, $y = 2$, $y = 2x-3$.

25. Find the coordinates of the vertices of the triangle whose sides are $5y = 3x+4$, $y+4x = 10$, $6y+x+9 = 0$.

26. Two of the three points $(-1, 2)$, $(2, 4)$, $(4, 5)$ are on the line $3x-5y+13 = 0$. Which are they?

**1.7.** We now take up the problem of finding the equation of a line when we know certain geometrical facts about it.

EXAMPLE 1. Find the equation of a line of gradient 3 passing through the point $(-2, 3)$.

*First solution.* The equation of the family of lines with gradient 3 is $y = 3x+k$.

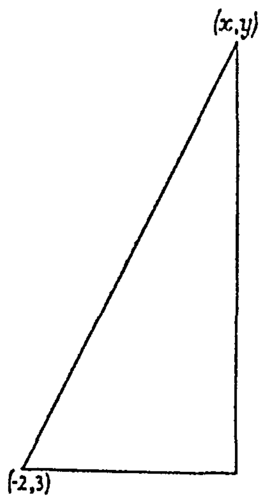
The line whose equation we have to find passes through $(-2, 3)$. Therefore we must choose $k$ so that the equation of the line is satisfied when $x = -2$, $y = 3$. Therefore $3 = 3(-2)+k$ and $k = 9$.

The equation of the line is $y = 3x+9$.

*Second solution.* Let $(x, y)$ be any point of the line. Then, from a sketch, we see that the gradient of the line is $(y-3)/\{x-(-2)\}$ and must $= 3$.

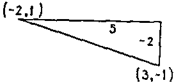
Therefore the equation of the line is

$$\frac{y-3}{x+2} = 3,$$
$$y-3 = 3x+6,$$
$$y = 3x+9.$$



EXAMPLE 2 Find the equation of the line joining the points $(-2, 1)$ and $(3, -1)$

*Solution* From a sketch we see that when $x$ increases by 5, $y$ decreases by 2 Hence the gradient of the straight line joining the given points is $-\frac{2}{5}$ We now know the gradient and we can take either $(-2, 1)$ or $(3, -1)$ as a point on the line The problem is reduced to one like Example 1 The equation of the family of lines with gradient $-\frac{2}{5}$ is $y = -\frac{2}{5}x+k$




We have to choose $k$ so that $(-2, 1)$ is on the line Therefore $1 = \frac{4}{5}+k$

Therefore $k = \frac{1}{5}$ and the equation of the line is $y = -\frac{2}{5}x+\frac{1}{5}$, or $5y+2x = 1$

*Check* Verify that $(3, -1)$ is on the line

$$5(-1)+2(3) = 1$$

*To write down the equation of the line with gradient $-\frac{2}{5}$, passing through the point $(-2, 1)$*

*First method* Write down the equation of the family of parallel lines with gradient $-\frac{2}{5}$ in the form $5y+2x = k$ to avoid fractions Then the constant $k$ must be chosen so that the equation is satisfied when $x = -2$ and $y = 1$ Therefore $5-4 = k$, and $k = 1$

In practice all that need be written down is

$$5y+2x = 5(1)+2(-2) = 1$$

*Second method* If $(x, y)$ is any point on the line we have the following table of values

| | |
|---|---|
| $-2$ | $x$ |
| $1$ | $y$ |

Therefore the gradient of the line is $(y-1)/\{x-(-2)\}$ and must $= -\frac{2}{5}$

Hence the required equation is

$$\frac{y-1}{x+2} = -\frac{2}{5},$$
$$5y-5 = -2x-4,$$
$$5y+2x = 1$$

EXERCISE 1 F

1 Write down the equations of the families of parallel lines with the following gradients (*a*) 2, (*b*) $\frac{1}{2}$, (*c*) $-\frac{2}{3}$, (*d*) $-\frac{11}{7}$

2 Write down the equations of the following lines

(i) with gradient 4 passing through $(1, 1)$,

(ii) with gradient $-1$ passing through $(2, \frac{1}{2})$,

(iii) with gradient $\frac{3}{5}$ passing through $(3, 5)$,

(iv) with gradient $-\frac{3}{2}$ passing through $(1, -1)$,

(v) joining $(1, 2)$ and $(4, 5)$,

(vi) joining $(-1, -3)$ and $(2, 7)$;
(vii) joining $(2, 2)$ and $(5, 2)$.

3. What is the gradient of the line $2y+3x = 7$? Show that $(2, \frac{1}{2})$ is on this line and find the equation of a line through $(2, \frac{1}{2})$ which is equally inclined to the axes.

4. Find the equation of the line through $(3, -4)$ parallel to $3x+4y = 18$.

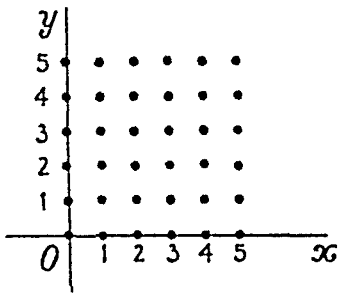
5. A portion of a playground was once marked out in squares to represent graph paper, and coordinate axes were marked. The boys of a class were then told to arrange themselves so that

(i) their $x$-coordinates were all 0;
(ii) their $y$-coordinates were all 0;
(iii) each boy's $x$-coordinate was equal to his $y$-coordinate.

Each time the boys found themselves on a straight line:

(*a*) describe each line; (*b*) give the equation of each line.

6. Close to the colour-light signals on the up lines to a London terminus, boards are set up each containing a square of 36 white lights. These can be illuminated to form numbers and so tell the driver of an incoming train the number of the platform he is running in to. If each light is defined by its coordinates referred to $x$- and $y$-axes in the lower and left-hand edges of the square, to what platform is a train signalled when the illuminated lights are $(0, 2)$, $(1, 2)$, $(2, 2)$, $(3, 2)$, $(4, 2)$, $(3, 0)$, $(3, 1)$, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(1, 3)$, $(2, 4)$? Find the equations of the three lines of which the number is formed.




7. What is the direction angle of the line joining $(1, 3)$ and $(1, -5)$? Write down the equation of the line.

8. Find the equations of the lines through $(7, -8)$ parallel to the coordinate axes.

9. Find the equations of the lines through $(1, 2)$ making equal angles with the coordinate axes.

10. Find the equation of the line joining $(-3, -1\frac{1}{2})$ and $(2, 2\frac{1}{2})$. Are the points $(-3, -1\frac{1}{2})$, $(2, 2\frac{1}{2})$, $(10, 8{\cdot}9)$ on a straight line?

11. Find the equation of the reflection of $2x-y = 4$ (*a*) in the $x$-axis, (*b*) in the $y$-axis. Verify that the two reflections are parallel.

12. Find the equation of the line through $(1, 2)$ with gradient $-\frac{1}{2}$. Find the equation of the reflection of this line in the $x$-axis. Verify that the reflection of $(1, 2)$ is on it.

13 Find the equations of the lines through $(-2, -4)$ whose direction angles are $45°$ and $135°$

14 $ABCD$ is a parallelogram $A \equiv (-2, -1)$, $B \equiv (1, 0)$, $C \equiv (2, 2)$ Find the equations of $AD$ and $CD$ and the coordinates of $D$

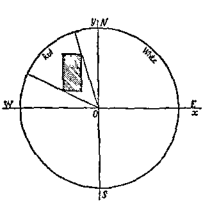
15 An observation post is set up and axes are laid down with origin at the observation post, positive $x$ axis towards the east and positive $y$ axis towards the north It is agreed that the scale shall be such that the points $(0, 1)$, $(1, 0)$ are each 1 mile from the origin A straight railway line is reported with a station at $(4, 1)$ and the entrance to a tunnel at $(-2, 3)$ Find the equation of the railway A road bridge is reported over the railway the line of sight from the observer to the bridge being $y = 2x$ Define the position of the bridge and plot it Apply Pythagoras' Theorem to show that the bridge is $\sqrt{10}$ *miles from the station and find its distance from the tunnel*

The following objects are sighted in the positions given Plot their positions and say whether they are on the nearer or farther side of the railway (a) a pond at $(-1, 2)$, (b) a haystack at $(3, 1\ 6)$

A farm house is situated at $(0, -2)$ What is the bearing from the farm house to the nearest degree, of (a) the station? (b) the tunnel?

16 An origin is taken at the Owe lighthouse and axes are laid down, the positive $x$ axis towards the east and the positive $y$ axis towards the north The unit of distance on both axes is 1 mile

The lighthouse covers with a red sector a dangerous shoal known, from its shape, as the Oblong Bank If the corners of the shoal are at $(-1, 1)$, $(-1, 3)$, $(-2\ 3)$, $(-2, 1)$, find the equations of the limits of the red sector What is its angle? The track of a steamer is $2y+x = 7$ A trawler at $(3, 4)$ is steering the same number of degrees S of W as the steamer is N of W Find the equation of the track of the trawler Find the coordinates of the points in which the vessels enter the red sector The lighthouse keeper has orders to fire a gun when a vessel is standing in to danger For which vessel should the gun be fired?



**1.8.** The following facts about the application of algebra to the solution of problems in geometry have emerged in Exercises 1.E and 1.F and deserve special notice.

(1) A point is on a straight line if its coordinates satisfy the equation of the line and, of course, is *not* on the straight line if its coordinates do *not* satisfy the equation.

(2) The coordinates of the point of intersection of two lines are found by solving the equations of the lines as simultaneous equations.

(3) The equation of a line parallel to the $x$-axis is of the form $y = k$. The equation of a line parallel to the $y$-axis is of the form $x = k$. In particular the equation of the $x$-axis is $y = 0$ and the equation of the $y$-axis is $x = 0$. Hence the points in which a line meets the axes are found by putting $x = 0$, $y = 0$, in turn, in the equation of the line.

The points in which a line cuts the axes are particularly useful because they can be plotted so easily.

EXERCISE 1.G

Find the coordinates of the points in which the following lines cut the axes. Hence sketch the lines.

1. $2x+y = 4$.
2. $3x-2y = 12$.
3. $x+y+1 = 0$.
4. $2x-4y+4 = 0$.
5. $x/2+y/3 = 1$.

## 1.9. Intercepts

A line cuts the axes in the points $(5, 0)$, $(0, -7)$. 5 and $-7$ are called the *intercepts* of the line on the axes. The intercept on the $x$-axis is always named before the intercept on the $y$-axis. Note that each intercept is a directed length.

The line with intercepts 5 and $-7$ has gradient $\frac{7}{5}$, and it passes through $(5, 0)$. Hence its equation is

$$5y-7x = 5\times 0-7\times 5 = -35.$$

Dividing by $-35$ this can be written

$$\frac{x}{5}-\frac{y}{7} = 1 \quad \text{or} \quad \frac{x}{5}+\frac{y}{-7} = 1.$$

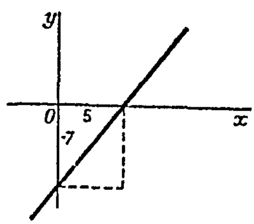


FIG. 1.14

In this form the intercepts can be read from the equation immediately. Conversely the equation of a line making given intercepts on the axes can be written down straight away.

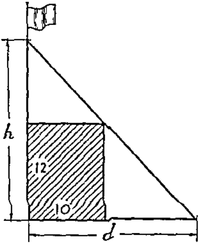
e g (1) if the intercepts are 1, 2, the equation of the line is $x/1+y/2 = 1$

(2) if the intercepts are $-2$, 3, the equation of the line is $x/(-2)+y/3 = 1$

[Put $x = 0$ in (1) and (2) What do you find for $y$? Put $y = 0$ and determine $x$ This checks that the equations written down make the stated intercepts on the axes ]

In general if $a$ and $b$ are constants the equation $x/a+y/b = 1$ represents a straight line making intercepts $a$, $b$ on the $x$ and $y$ axes respectively

EXERCISE 1 H




Ex 1 H 8

1 Write down the equation of the line making intercepts of 3 and 4 on the axes Check by finding the gradient of the line and hence obtaining the equation

2–6 Write down the equations of the lines making the following intercepts on the axes

(2) 5, 4 (3) $-3$ 6

(4) $-2$, $-4$ (5) $\frac{1}{2}$, $\frac{1}{3}$

(6) $-\frac{1}{2}$, $\frac{2}{3}$

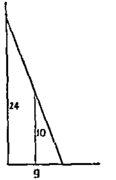
7 A line makes intercepts on the axes of 5 and 2 and passes through the point $(a, b)$ Show that $2a+5b = 10$

8 A straight wire is to be set up to secure a flagstaff but will have to pass over a shed whose dimensions are shown in the sketch If the wire just passes over a corner of the shed, show that

$$10/d+12/h = 1$$

(Choose your own axes )

Find $h$ if $d = 25$



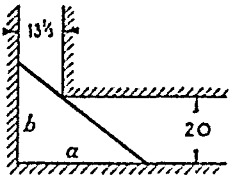

Ex. 1 H 9

9 A ladder is in contact with a wall 24 ft up from the ground and with the ground 9 ft out from the wall What is the distance from the wall of a man whose feet are on a rung of the ladder 10 ft above the ground? (Choose your own axes )

10 Find the equation of the reflection of $x/5-y/2 = 1$ (a) in the $x$ axis, (b) in the $y$ axis

11. What is the equation of the family of lines making intercepts on the axes in the ratio 3:4? [Call the intercepts of any line $3k$, $4k$.]

12. The diagram shows the predicament of two men trying to manœuvre a ladder from a lane $13\frac{1}{3}$ ft. wide to one 20 ft. wide. Both lanes are bounded by high walls. The ladder jams when $a = 40$. Find $b$ and the length of the ladder.




Ex. 1.H 12

13. What is the gradient of the straight line which makes intercepts $a$ and $b$ on the axes? Hence prove that its equation is

$$x/a+y/b = 1.$$

## 1.10. The relation between the gradients of two perpendicular lines

Let $m_1$, $m_2$ be the gradients, and $\theta_1$, $\theta_2$ the direction angles of two perpendicular lines. Assume, for the moment, that neither line is parallel to a coordinate axis. Then $\theta_2 = 90+\theta_1$.

Therefore

$$\tan\theta_2 = \tan(90+\theta_1) = -(1/\tan\theta_1).$$

But $\tan\theta_2 = m_2$ and $\tan\theta_1 = m_1$.

Hence $$m_2 = -\frac{1}{m_1},$$

or $$m_1 m_2 = -1.$$

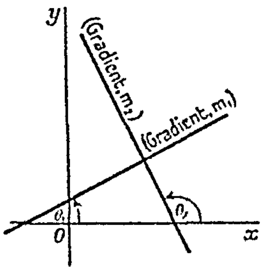
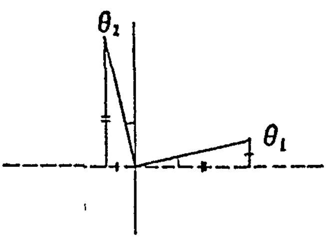



FIG. 1.15

There remains the case when one line is parallel to an axis. Then the other line is parallel to the other axis and the equations of the lines have the forms $x = c_1$ and $y = c_2$.

*If two lines are perpendicular they are either parallel to the axes or the product of their gradients is* $-1$.




$\tan\theta_1 = -1/\tan\theta_2$

EXAMPLE. Find the equation of the line through (5, 3) perpendicular to $2x-3y+5 = 0$.

*Solution.* The gradient of the given line is $\frac{2}{3}$. Therefore the gradient of any perpendicular line is $-\frac{3}{2}$.

The required equation is therefore

$$2y+3x = 2\times3+3\times5 = 21.$$

## Exercise I J

1 Find the direction angle of each of the lines $y-2x=1$, $2y+x=1$ Verify that they are perpendicular

2 Of the following three lines, two are perpendicular Which are they? (i) $3x-2y+7=0$, (ii) $2x-3y=5$, (iii) $2x+3y=1$

3 Find the equation of a line through the origin perpendicular to (a) $2y+x=0$, (b) $y-3x=0$, (c) $3y+5x=10$

4 Show that the following pairs of lines through the origin are perpendicular

(i) $2y-3x=0$, $3y+2x=0$, (ii) $5y+4x=0$, $4y=5x$,
(iii) $y-mx=0$, $my+x=0$

5 Find the equation of a line drawn through

(a) $(1, 3)$ perpendicular to $x+y+5=0$,
(b) $(-2, 2)$ perpendicular to $2x=3y+1$,
(c) $(1, -4)$ perpendicular to $5x+2y+3=0$,
(d) $(3, -2)$ perpendicular to $y+5=0$,
(e) $(4 \ -3)$ perpendicular to $x=1$

6 Find the equation of the perpendicular from the origin to $3y-7+2x=0$

7 Show that the lines joining (a) $(3, 4)$ and (b) $(-2, 1\frac{1}{2})$ to the origin are perpendicular Show that this is also true of the lines joining $(c, d)$, $(-d, c)$ to the origin

8 Show that the lines joining $(2, 3)$ to (a) $(-1, 0)$ and (b) $(5, 0)$ are perpendicular

9 If $A\equiv(0, 6)$, $B\equiv(0, -4)$, $C\equiv(4, -2)$, $D=(-3, 5)$ show that the angles $ACB$ and $ADB$ are right angles What can be said about the circle on $AB$ as diameter?

10 Two families of parallel slats are crossed at right angles to form a trellis The equation of one family is $y+x=c$ Find the equation of the other family

11 Find the equations of the diagonals of the quadrilateral whose vertices are $(3, 2)$, $(5, 1)$, $(3, -4)$, $(1, 1)$ and show that they are perpendicular

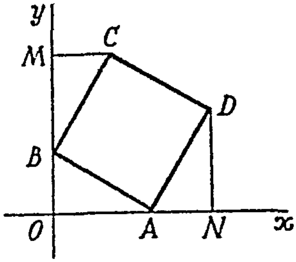
12 Show that the diagonals of the quadrilateral $(0, 3)$, $(4, 2)$, $(2, -1)$, $(0, 0)$ are perpendicular

13 Show that the lines joining $(1, 3)$ and $(-3, 1)$ to the origin are perpendicular If a square is drawn with these three points as three of its vertices find the equations of the four sides and the coordinates of the fourth vertex

14 *Find the equation of the lines through* $(1, 4)$ parallel and perpendicular to $3y=2x-1$

15. Find the equation of the line through $(-1, 3)$ perpendicular to $2y+x+4 = 0$ and the line through $(0, 0)$ parallel to this line. Find the coordinates of the vertices of the rectangle which has these three lines as three of its sides and $(0, 0)$ as one of its vertices.

16. $A \equiv (2, 0)$, $B \equiv (0, 1)$. A square $ABCD$ is drawn on $AB$ as side with its remaining vertices in the first quadrant




(i) Find the equation of $AB$.
(ii) If $DN$ is perpendicular to the $x$-axis and $CM$ is perpendicular to the $y$-axis, prove that the triangles $DAN$, $ABO$, $BCM$ are congruent.
(iii) Deduce from (ii) the coordinates of $C$ and $D$.
(iv) Find the equations of the diagonals of the square and verify that they are at right angles.

17. The vertices of a triangle are $A \equiv (0, 4)$, $B \equiv (-1, 0)$, $C \equiv (3, 0)$. Find the equations of the perpendicular from $B$ to $AC$ and from $C$ to $AB$. Show that they meet on the perpendicular from $A$ to $BC$. (It can be shown that the perpendiculars from the vertices of any triangle to the opposite sides meet at a point called the *orthocentre* of the triangle.)

## 1.11. The distance between two points whose coordinates are given

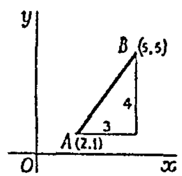



Fig. 1.16

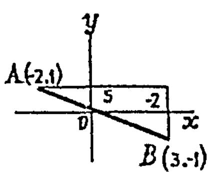



Fig. 1.17

If $A \equiv (2, 1)$ and $B \equiv (5, 5)$, $B$ may be reached from $A$ by a step of 3 parallel to the $x$-axis followed by a step of 4 parallel to the $y$-axis. Hence by the theorem of Pythagoras

$$AB^2 = 3^2+4^2 = 25.$$

The distance between the points is 5.

If $A \equiv (-2, 1)$ and $B \equiv (3, -1)$, $B$ may be reached from $A$ by an $x$-step of 5 $\{= 3-(-2)\}$ followed by a $y$-step of $-2$ $(= -1-1)$.

Therefore
$$AB^2 = 5^2+(-2)^2 = 29,$$
$$AB = \sqrt{29}.$$

Generally, if $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$, $B$ may be reached from $A$ by an $x$ step of $x_2-x_1$ followed by a $y$ step of $y_2-y_1$

Hence $AB^2 = (x_2-x_1)^2+(y_2-y_1)^2$

This formula need not be remembered but may easily be obtained, when required, from the following 'conventional sketch' which shows the points $A$ and $B$ not necessarily in their true positions but as if they were in the first quadrant

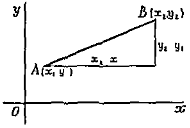


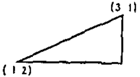

Fig 1 18

Note that such a sketch is also useful in finding the gradient of $AB$ For, as we have already seen, this gradient is found by dividing any change in $x$ into the corresponding change in $y$ In all cases, therefore, it is $(y_2-y_1)/(x_2-x_1)$

Example $A \equiv (-1, 2)$, $B \equiv (3, -1)$ Find (i) the distance, $AB$, (ii) the gradient of the line $AB$

*Solution* (i)

$$AB^2 = (3+1)^2+(-1-2)^2$$
$$= 4^2+3^2$$
$$= 25,$$
$$AB = 5$$




(ii) the gradient of $AB = \dfrac{-1-2}{3+1}$

$$= -\tfrac{3}{4}$$

## Exercise 1 K

1–2 Give the $x$ step and $y$ step by which $B$ is reached from $A$ and find the distance $AB$ and the gradient of $AB$

1 $A \equiv (-2, 1)$, $B \equiv (6, 7)$

2 $A \equiv (-3, 2)$, $B \equiv (3, -4)$

3 Show that the triangle whose vertices are $A \equiv (1, 3)$, $B \equiv (3, 4)$, $C \equiv (0, 1)$ is isosceles and name the equal sides

4 Show that the triangle whose vertices are $(0, 2)$, $(\sqrt{3}, -1)$, $(-\sqrt{3}, -1)$ is equilateral

5 If $A \equiv (1, 2)$, $B \equiv (5, 4)$, $C \equiv (7, -1)$, $D \equiv (2, -2)$, prove that the diagonals of the quadrilateral $ABCD$ are equal

6 If $A \equiv (-2, 3)$, $B \equiv (3, 4)$, $C \equiv (4, -1)$, $D \equiv (-1, -2)$, prove that $ABCD$ is a rhombus

7 Show that $(1, 2)$ is the middle point of the line joining $(-3, 5)$ and $(5, -1)$

## 1.12. The middle point of the straight line joining two given points

Let $M$ be the middle point of $AB$ where $A \equiv (1, 2), B \equiv (5, 3)$. To find the coordinates of $M$ we complete the right-angled triangles $APB$, $AQM$ by drawing lines parallel to the axes. Then

$$QM = \tfrac{1}{2}PB$$
$$= \tfrac{1}{2}(3-2)$$
$$= \tfrac{1}{2}.$$

Therefore the $y$-coordinate of $M$

$$= y\text{-coordinate of } A + \tfrac{1}{2}$$
$$= 2\tfrac{1}{2}.$$

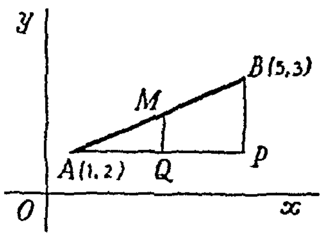



FIG. 1.19

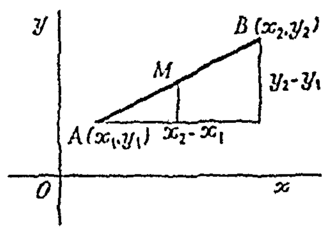



FIG. 1.20

Similarly $AQ = \tfrac{1}{2}AP = \tfrac{1}{2}.4 = 2$ and the $x$-coordinate of $M$

$$= \text{the } x\text{-coordinate of } A + 2$$
$$= 3.$$

$$M \equiv (3, 2\tfrac{1}{2}).$$

[As a check show that the equation of $AB$ is $4y - x = 7$ and verify that $M$ is on this.]

*Note that 3 is the arithmetic mean of the $x$-coordinates of* $A$ and $B$ and $2\tfrac{1}{2}$ is the arithmetic mean of the $y$-coordinates. This may be proved generally.

Let $\quad A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$.

$B$ is reached from $A$ by an $x$-step of $(x_2 - x_1)$ followed by a $y$-step of $(y_2 - y_1)$. Hence $M$ is reached from $A$ by an $x$-step of $(x_2 - x_1)/2$ followed by a $y$-step of $(y_2 - y_1)/2$.

The $x$-coordinate of $M = x_1 + \dfrac{x_2 - x_1}{2} = \dfrac{x_1 + x_2}{2}$.

The $y$-coordinate of $M = y_1 + \dfrac{y_2 - y_1}{2} = \dfrac{y_1 + y_2}{2}$.

Therefore $\quad M \equiv \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

## Exercise 1 L

1–5 Find the middle points of the lines joining the given points

1 $(0, 1), (0, 5)$ 2 $(-2, 0), (4, 0)$

3 $(1, 3), (7, 5)$ 4 $(-8, -2), (3, -4)$

5 $(10, -7), (-13, 9)$

6 $A = (2, -3), B = (4, -7)$ Show that $2y-x+13 = 0$ is the perpendicular bisector of $AB$

7–8 Find the equation of the perpendicular bisector of $AB$

7 $A = (2, 4), B = (4, 6)$ 8 $A = (-1, 1\frac{1}{2}), B = (3, -3\frac{1}{2})$

## Exercise 1 M

1 The sides of a square are $y = 2$, $y+1 = 0$, $x = 1$, $x = 4$ Verify that the middle points of the diagonals coincide Verify that the diagonals are at right angles

2 Two wireless masts are 28 ft and 16 ft high and the aerial is stretched taut between their tops Find the height of the middle point of the wire

3 Find the equations of the lines through $(1, 0)$ and $(-1, 0)$ parallel to $3y+2x = 0$ Let the intersections of these lines with $y+x = 2$ be $A, B, C$ respectively Find the coordinates of $A, B, C$, and show that $AC = CB$ Of what geometrical theorem is this an example?

4 Find the gradients of the diagonals of the rectangle whose vertices are $(-2, 4), (6, 4), (6, -1), (-2, -1)$ Show that the diagonals are equally inclined to the axes and equal in length

5 Find the equations of the sides of a rectangle of which $(0, 0)$ and $(1, 7)$ are opposite vertices and $3y-x = 0$ is one side Find the coordinates of the remaining two vertices

6 $A = (2, 5), B = (-1, -1), C = (6, 1)$ Find the coordinates of a point $D$ such that $ABCD$ is a parallelogram (Use the fact that the diagonals bisect each other You need not find the equations of the sides)

Check by verifying that $AB = CD$, $AD = BC$

7 $A = (10, 16), B = (-2, 3), C = (6, -1)$ $D$ is the middle point of $BC$ Find the length of $AD$

8 A triangle $ABC$ has vertices $A = (1, 4)$, $B = (-2, -3)$, $C = (3, -1)$ Find the coordinates of the middle points of $AB$ and $BC$ Verify that their join is parallel to $AC$

9 $ABCD$ is the quadrilateral $(-2, 2), (3, 4), (5, -2), (-4, -3)$ $M, N, P, Q, X, Y$, are the middle points of $AB, CD, BC, DA, AC, BD$ Find the coordinates of the middle points of $XY, MN, PQ$ and show that they coincide

10. $ABCO$ is a trapezium such that $A \equiv (0, 3)$, $B \equiv (4, 3)$, $C \equiv (8, 0)$, $O \equiv (0, 0)$. If $D$, $E$ are the middle points of $AC$, $BO$, find the coordinates of the middle point, $X$, of $ED$.

If $Y$ is the intersection of $AC$, $BO$ find the coordinates of $Y$ and the equation of $XY$. Show that $XY$ meets $AB$ and $OC$ at their middle points.

11. Show that $(7, -1)$, $(2, 4)$, $(6, 2)$, $(5, 3)$ are all the same distance from $(2, -1)$ and find the distance.

12. Show that the points $(2, 2)$, $(5, 3)$, $(6, 0)$ lie on a circle with centre $(4, 1)$. Find its radius. Also find the coordinates of the points in which $x = 3$ cuts this circle.

## 1.13. Loci

Consider the straight line $y = x+1$. This equation enables us to select from all the points in the plane those that lie on the straight line. For example, we see that the points $(1, 2)$, $(-1, 0)$, $(2, 3)$, $(4, 5)$ lie on the line because their coordinates satisfy the equation of the line. On the other hand, $(2, 1)$ does not lie on the line for $1 \neq 2+1$.

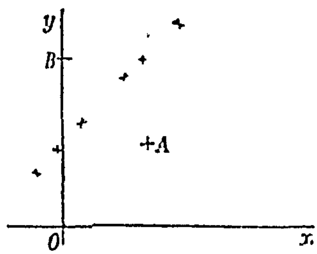


FIG. 1.21

Now suppose we are asked to select a point $P$ in the plane of the axes equidistant from $A \equiv (1, 1)$ and $B \equiv (0, 2)$. With a pair of compasses we quickly see that we are free to choose any one of a large number of points.

Let $(x, y)$ be the coordinates of *any* point, $P$, satisfying the given condition. Then $AP^2 = BP^2$.

Therefore
$$(x-1)^2+(y-1)^2 = x^2+(y-2)^2,$$
$$x^2-2x+1+y^2-2y+1 = x^2+y^2-4y+4,$$
$$-2x-2y+2 = -4y+4,$$
$$2y = 2x+2,$$
$$y = x+1.$$

Every point $P \equiv (x, y)$ which is equidistant from $A$ and $B$ must have coordinates which satisfy this equation. Hence all the points in the plane which are equidistant from $A$ and $B$ lie on the straight line $y = x+1$.

This straight line is called the *locus* of the points which are equidistant from $A$ and $B$.

You may know from elementary geometry that this locus is the perpendicular bisector of $AB$. If not, this is easily verified.

The mid point of $AB$ is $(\frac{1}{2}, \frac{3}{2})$ The gradient of $AB$ is $-1$ Hence the gradient of the perpendicular bisector of $AB$ is $+1$ and its equation is

$$y-x = \tfrac{3}{2}-\tfrac{1}{2} = 1,$$

or

$$y = x+1$$

## Exercise 1 N

1–4 Find the equation of the locus of the points which are equidistant from

1 $(-1, 0), (3, 0)$

2 $(0, 8), (4, 0)$

3 $(-1, 3), (3, -1)$

4 $(-1, -5), (3, 3)$

5 Write down the equation of the locus of a point which may be reached from a point on the $x$ axis by a step of $+2$ parallel to the $y$ axis

6 Write down the equation of the locus of a point which may be reached from a point of the $y$ axis by a step of $-3$ parallel to the $x$ axis

7 $P$ is a point in the first or third quadrant such that its distance from the $x$ axis is twice its distance from the $y$ axis What is the equation of its locus ?

8 Three times the $x$ coordinate of $P$ together with five times its $y$ coordinate is always 18 What is the equation of the locus of $P$ ?

9 The distance of $P$ from the $x$ axis equals its distance from the $y$ axis Write down the equations of the two straight lines which together make up the locus of $P$

10 The square of the distance of $P$ from the $x$ axis equals four times its distance from the $y$ axis Write down the equation of its locus, plot a few points, and sketch the curve

11 $A \equiv (1, 0)$, $B \equiv (-1, 0)$ $P \equiv (x, y)$ is a point such that $\angle APB$ is a right angle Find the equation of the locus of $P$ [From elementary geometry we can recognize this locus as a circle on $AB$ as diameter ]

12 Write down the equation of the locus of a point such that the square of its distance from the origin is 2

13 Write down the equation of the locus of a point whose distance from the origin is 5

14 What is the equation of the locus of a point whose distance from the point $(1, 2)$ is always 2 ?

15 What are the names of the curves whose equations are (a) $x^2+y^2 = 100$, (b) $(x-1)^2+(y-2)^2 = 4$

## 1.14. The equation of a circle

If the point $(x, y)$ is on a circle with centre $(-1, 3)$ and radius 3, its distance from $(-1, 3)$ is always 3. Hence

$$(x+1)^2+(y-3)^2 = 3^2$$

is the equation of the circle.

This simplifies to

$$x^2+y^2+2x-6y+1 = 0.$$

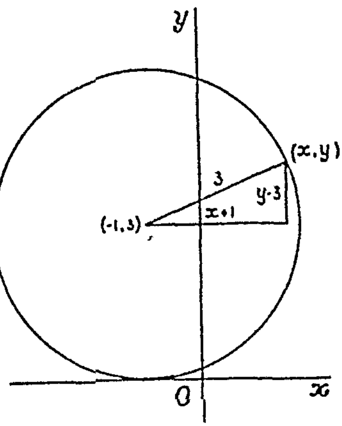



Fig. 1.22

Conversely, an equation of the type $x^2+y^2-4x+8y+4 = 0$ represents a circle.

To show this we complete the squares of the terms in $x$ and the terms in $y$.

$$x^2-4x+4+y^2+8y+16 = 16,$$
$$(x-2)^2+(y+4)^2 = 4^2.$$

This expresses the fact that the distance of $(x, y)$ from $(2, -4)$ is always 4. Hence the given equation represents a circle of centre $(2, -4)$ and radius 4.

Note that a circle has an equation which is of the second degree.

### Exercise 1.P

1–5. Find the centre and radius of the circles whose equations are:

1. $x^2+y^2-2y-3 = 0.$
2. $x^2+y^2-2x-2y-7 = 0.$
3. $x^2+y^2+2x-6y+9 = 0.$
4. $x^2+y^2 = 36.$
5. $x^2+y^2+2x+10y+22 = 0.$

6–10. Which of these equations represent circles?

6. $x^2+y^2-4 = 0.$
7. $x^2+2x+4y = 0.$
8. $x^2+xy+y^2 = 16.$
9. $2x^2+y^2+4x+2y-46 = 0.$
10. $2x^2+2y^2-7x+8y = 0.$

11–13. Write down and simplify the equations of the circles:

11. With centre $(0, 2)$, radius 2.
12. With centre $(1, -1)$, radius 3.
13. With centre $(-1, -2)$, radius $1\frac{1}{2}$.

14. Find the coordinates of the points of intersection of the circles $x^2+y^2 = 5$ and $x^2+y^2+2x = 7$.

15 Find the coordinates of the two points on $7y+x = 25$ whose distance from the origin is 5

16 $A \equiv (-2, 0)$, $B \equiv (4, 0)$ $P$ is a point such that $2PA = PB$ Find the equation of the locus of $P$ Identify it as a circle and draw a figure

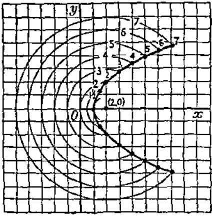
## 1.15. Further loci

Other curves whose equations are of the second or higher degree frequently appear as loci

The following exercise introduces some more second degree loci

EXERCISE 1 Q

1 The distance of a point $P$ from $(2, 0)$ equals its perpendicular distance from the $y$ axis Find the equation of the locus of $P$ [Hint The use of a square root may be avoided by noting that if the distances are equal so are their squares ]




[This locus may easily be drawn on graph paper with the help of a pair of compasses Take the side of 1 square of the paper as the unit of length With centre (2, 0) draw circles of radii 1, $1\frac{1}{2}$, 2, 3, 4, 5, 6, 7 The intersections of these circles with the lines

$x = 1,\ x = 1\frac{1}{2},\ x = 2,\ x = 3,\ x = 4,\ x = 5,\ x = 6,\ x = 7$ ,

are points on the locus Note that the construction shows that it is symmetrical about the $x$ axis ]

2 A point moves so that its distance from $(0, 2)$ equals its perpendicular distance from the $x$ axis Find the equation of the locus

3 A point moves so that its distance from $(1, 0)$ equals its perpendicular distance from $x+1 = 0$ Find the equation of its locus

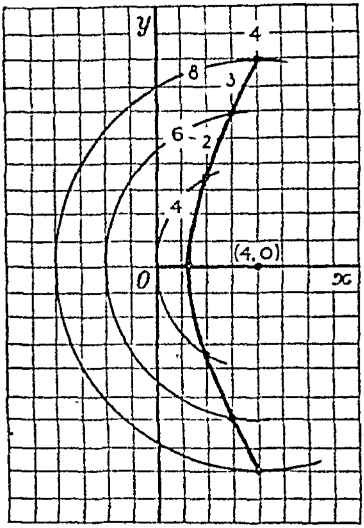
4 A point moves so that its distance from $(0, \frac{1}{4})$ equals its distance from $y+\frac{1}{4} = 0$ Find the equation of the locus

[Though the equations of the loci obtained in Nos 1, 2, 3, and 4 differ considerably, the law governing the positions of the points of each locus is of the same type The differences are caused by a different choice of axes, fixed point, and fixed line The curves of Nos 1–4 are all called *Parabolas* ]

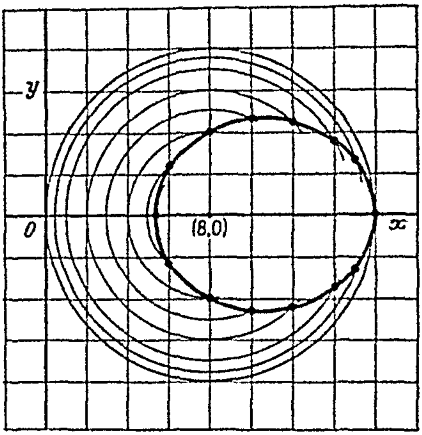
5. The distance of a variable point $P$ from $(4, 0)$ is twice its perpendicular distance from the $y$-axis. Find the equation of the locus of $P$.

[The method suggested for drawing the locus of No. 1 may be adapted to draw this locus. It is not a parabola.]

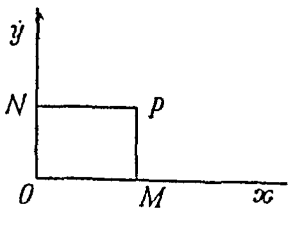
6. The distance of a variable point $P$ from $(8, 0)$ is half its perpendicular distance from the $y$-axis. Find the equation of the locus of $P$. This may be drawn by adapting the method suggested for No. 1.



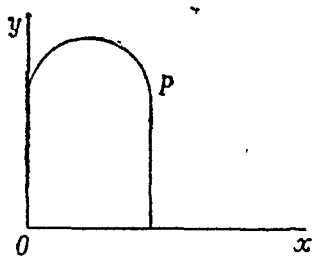

Ex. 1.Q 5




Ex. 1.Q 6




Ex. 1.Q 7




Ex. 1.Q 8

7. The point $P$ is taken in the first quadrant so that the area of the rectangle $OMPN$ is constant and equal to $a^2$. Find the equation of the locus of $P$. Draw the curve for $a = 2$ from $x = \frac{1}{2}$ to $x = 8$.

8. An architect wishes to design a doorway surmounted by a semicircular arch to have a total area of 36 sq. ft. He makes various trial sketches. If these sketches are placed so that the bottom and left-hand edges lie along the axes shown (unit on each axis: 1 ft.), find the equation of a curve passing through all the points like $P$.

## 1 16 The general linear equation

All linear equations may be obtained from $ax+by+c = 0$ by giving $a$ $b$ $c$ suitable values Thus $3x+2\ 7y-50 = 0$ is obtained by putting $a = 3$ $b = 2\ 7$ $c = -50$ $ax+by+c = 0$ is called the *general* linear equation and $a$ $b$ $c$ are called the *coefficients* of the equation

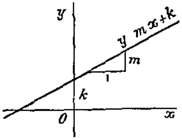



Fig 1 ·3

If any values are given to $a$ $b$ $c$ (provided that both $a$ and $b$ are not 0) we obtain the equation of a straight line The equation of a straight line may be altered by dividing through by any non zero coefficient without changing the straight line which it represents $3x-2y+5 = 0$ and $x-\frac{2}{3}y+\frac{5}{3} = 0$ represent the same straight line

If $b \neq 0$ the general equation may be written

$$y = -\frac{a}{b}x-\frac{c}{b}$$

$$= mx+k$$

where $m = -a/b$ and $k = -c/b$

Then $m$ and $k$ have simple geometrical meanings $m$ is the gradient of the line and $k$ is the intercept on the $y$ axis If however $b = 0$ the general equation becomes $ax+c = 0$ or $x = -c/a$ ($a$ cannot also be 0) Writing $d = -c/a$ this equation becomes $x = d$

The locus of a point whose $x$ coordinate is fixed is a straight line parallel to the $y$ axis Hence for different values of $d$ $x = d$ represents the family of lines parallel to the $y$ axis These lines have no gradient and their direction must be defined by their direction angles which are all 90° [Try to find a number $a$ so that you can say that $x = d$ rises $a$ in 1 You cannot Try to put the equation in the form $y = mx+d$ so that you can read off the gradient You cannot ]

Note that the equation $y = mx+c$ is the general equation of all lines not parallel to the $y$ axis while $ax+by+c = 0$ is the general equation of all lines in the plane of the axes

## Exercise 1.R

1. Find the value of $c$ if $ax+by+c = 0$ passes through the origin.

2. What can you say about the line $ax+by+c = 0$ if (i) $a = 0$, (ii) $b = 0$, (iii) $c = 0$, (iv) $b = c = 0$?

3. Write down the equations of the following families of lines:
   (a) the lines parallel to the $x$-axis;
   (b) the lines parallel to the $y$-axis;
   (c) the lines through the origin.

4. Find the equations of the lines of the following families which pass through the origin:
   (i) $y = 5x+k$; (ii) $3x+2y = c$.

5. Using tables, find approximately the gradient of a line whose direction angle is (a) 88°, (b) 89°, (c) 89° 54′.

6. Write down the equation of the line making intercepts $a$ and $b$ on the axes.

7. Choose $k$ so that the lines $y = mx+k$ pass through the point (0, 2).

8. Choose $k$ so that $y = 3x+k$ passes through (1, 3). Show that the line also passes through the origin.

9. Choose $m$, $k$ so that $y = mx+k$ makes an intercept of 4 on the $y$-axis and has gradient 2.

10. Choose $m$, $k$ so that $y = mx+k$ makes an intercept $b$ on the $y$-axis and has gradient $g$.

11. The observations of an experiment when plotted on a graph give the points (0, 1·8), (1·0, 2·5), (3·2, 3·8), (5·1, 5·0), (7·0, 6·1). Plot the points. They are approximately on a straight line. Find the equation of the straight line. This gives the law connecting the two quantities plotted.

12. I find in a catalogue the following prices for galvanized anchors:

| Weight (lb.) ($W$) | 50 | 65 | 80 | 90 |
|---|---|---|---|---|
| price (shillings) ($P$) | 40 | 52 | 64 | 72 |

Draw a graph and find the law giving $P$ in terms of $W$.

13. Smaller anchors of a different pattern are priced as follows:

| Weight (lb.) ($W$) | 20 | 30 | 40 |
|---|---|---|---|
| Price (shillings) ($P$) | 22 | 28 | 34 |

Find the law giving $P$ in terms of $W$. A 25-lb. anchor costs 26*s.* and a 35-lb. anchor costs 30*s.* Do these figures exactly fit the law you have found?

14 The following table gives the average weights of men of different heights †

| | | | | | |
|---|---|---|---|---|---|
| $x$ = Height (in ) | 62 5 | 64 5 | 66 5 | 69 5 | 70 5 |
| $y$ = Weight (lb ) | 133 | 139 | 146 | 156 | 160 |

Plot these figures and show that they fall approximately on the straight line joining the first and last points Writing the equation of this line in the form $y = mx+c$, determine $m$ to one decimal place and $c$ to the nearest unit What is the law giving the weight $W$lb of a man of height $h$ in ?

[The fact that this law is approximately linear is interesting It shows that men of different heights are not geometrically similar For assuming their densities to be approximately constant, if they were similar we should expect their weights to vary as the cubes of their heights There is a tendency for tall men to be lean, and short men to be stocky ]

## 1.17. The general equation of the circle

If $P \equiv (x, y)$ is a point on a circle with centre $C \equiv (1, 2)$ and radius $2\frac{1}{2}$, the equation of the circle is found by expressing the fact that $CP^2 = \frac{25}{4}$ in terms of the coordinates of $P$ and $C$ Hence the equation is

$$(x-1)^2+(y-2)^2 = \tfrac{25}{4},$$
$$x^2-2x+1+y^2-4y+4 = \tfrac{25}{4},$$
$$x^2+y^2-2x-4y-\tfrac{5}{4} = 0,$$
$$4x^2+4y^2-8x-16y-5 = 0$$

This equation is of the form

$$a(x^2+y^2)+bx+cy+d = 0,$$

where $a, b, c, d$ are constants, and in this case have the values $a = 4$ $b = -8$ $c = -16$, $d = -5$ It is clear that whatever point is taken as the centre of the circle and whatever the value of its radius the equation of the curve is always of this form, i e it is of the second degree and has its second-degree terms in the special form $ax^2+ay^2$ $(a \neq 0)$

Thus the equation of every circle is of the form

$$ax^2+ay^2+bx+cy+d = 0 \quad (a \neq 0)$$

We now show that every equation of the form

$$ax^2+ay^2+bx+cy+d = 0 \quad (a \neq 0)$$

† Selected from data given by Sir D Arcy Thompson, *Growth and Form*, p 203 It is only fair to add that the selection consists in taking the figures that fit the linear law best!

either represents a circle or no curve at all. Since $a \neq 0$, we write the equation in the form

$$x^2+y^2+\frac{b}{a}x+\frac{c}{a}y+\frac{d}{a} = 0,$$

or, writing $2g$ for $b/a$, $2f$ for $c/a$ and $k$ for $d/a$,

$$x^2+y^2+2gx+2fy+k = 0.\dagger$$

Completing the squares of the terms in $x$ and the terms in $y$, we have
$$(x+g)^2+(y+f)^2+k = g^2+f^2.$$

Therefore
$$(x+g)^2+(y+f)^2 = g^2+f^2-k. \tag{1}$$

If $(x, y) \equiv P$ and $(-g, -f) \equiv C$, this equation shows that

$$CP^2 = g^2+f^2-k.$$

Hence if $g^2+f^2 > k$, $CP = \sqrt{(g^2+f^2-k)}$ and $P$ is at a constant distance from the fixed point $C$. Hence the locus of $P$ is a circle of radius $\sqrt{(g^2+f^2-k)}$ and centre $(-g, -f)$.

If $g^2+f^2 < k$, the right side of equation 1 is negative. But the left side is the sum of two squares and is therefore positive for every pair of values of $x$ and $y$. Hence, in this case, equation 1 cannot be satisfied by the coordinates of any point, so far as we can see at present.

The equation
$$x^2+y^2+2gx+2fy+k = 0,$$

is called the *general equation of the circle*. The equation of any circle can be written in this form, and every such equation either represents a circle or no curve at all. Hence, if a given equation of the second degree represents a locus, this locus is a circle provided that the two following conditions are satisfied:

(1) coefficient of $xy = 0$;
(2) coefficient of $x^2$ = coefficient of $y^2$.

The general equation of the circle is used, chiefly, when we want to find the equation of a circle satisfying given geometrical conditions.

EXAMPLE 1. Find the equation of the circle passing through (1, 1), (4, 2), (−1, 7).

*Solution.* The required equation is $x^2+y^2+2gx+2fy+k = 0$ with suitable values of $g, f, k$.

Since (1, 1) is on the circle $1+1+2g+2f+k = 0$,
,, (4, 2) ,, ,, ,, $16+4+8g+4f+k = 0$,
,, (−1, 7) ,, ,, ,, $1+49-2g+14f+k = 0$.

† The unalphabetical choice here is made in anticipation of later work which is outside the scope of this book.

These equations reduce to
$$2+2g+2f+k=0,$$
$$20+8g+4f+k=0,$$
$$50-2g+14f+k=0,$$

and we can solve them as simultaneous equations obtaining
$$f=-4\tfrac{1}{2},\ g=-1\tfrac{1}{2},\ k=10$$

The equation of the circle is therefore
$$x^2+y^2-3x-9y+10=0$$

EXAMPLE 2 Find the equation of the circle cutting the $x$ axis in $(-1, 0)$, $(2, 0)$ and passing through $(1, 4)$

*Solution* This example can be solved by the method of the previous example The general equation may, however, be used to somewhat greater advantage as follows

In the general equation $x^2+y^2+2gx+2fy+k=0$ put $y=0$ Then $x^2+2gx+k=0$ gives the $x$ coordinates of its intersections with the $x$ axis But these are known to be $x=-1$, $x=2$, and the quadratic equation giving these solutions must be $(x+1)(x-2)=0$, i e $x^2-x-2=0$ Hence the circle will cut the $x$ axis in the required points if we take $2g=-1$, $k=2$

Accepting this suggestion, we set out the work as follows

The equation $x^2+y^2-x+2fy-2=0$ represents a circle cutting the $x$ axis in $x=1$, $x=2$

If the circle passes through $(1, 4)$
$$1+16-1+8f-2=0,$$
$$f=-\tfrac{7}{4}$$

The required circle is therefore
$$x^2+y^2-x-\tfrac{7}{2}y-2=0,$$
or
$$2x^2+2y^2-2x-7y-4=0$$

EXERCISE 1 S

1 Find the equation of the circle through the origin and $(0, 1)$, $(1, 0)$ Find the centre and radius of the circle and illustrate the results by a sketch

2 Find the equation of the circle through $(0, 0)$, $(2, 4)$, $(-3, 9)$

3 What is the value of $f$ in the general equation if the centre of the circle is on the $x$ axis ?

4 What do you know about the circle if (a) $g=0$, (b) $k=0$ ?

5 Find the equation of the circle through $(0, 0)$, $(1, 3)$, $(5, -5)$ Find its centre and radius

6 Find the equation of the circle passing through $(0, 1)$, $(0, 3)$, and $(3, 0)$ Find the other point of intersection of the circle with the $x$ axis

7 Find the equation of the circle on $(-1, -1)$ and $(3, 3)$ as diameter

8. Show that (7, 0) lies on the circle through (0, 0), (5, −5), (2, 2).

9. Find the equations of the circles of radius 5 passing through (3, 0) and (−3, 0).

10. Find the equation of the circle through (2, 0), (6, 0), (0, 3) and find its other intersection with the $y$-axis.

11. Find the equation of the circle whose centre is at the origin and which meets $2y = x+2$ where $y = 0$. Also find the coordinates of its other intersection with this line.

12. Find the equations of the circles which pass through (0, 0), (1, −1) and have radius $\sqrt{5}$.

13. Find the centre and radius of the circle $4x^2+4y^2-16x-8y = 5$.

14. For what value of $a$ does the equation $x^2+ay^2 = 1$ represent a circle ?

15. The equation of a family of curves is $x^2+2ax+a^2y^2/4 = 0$. Find the equation of the circle belonging to this family.

16. Can you choose $a$ so that the following equations represent circles ?

(a) $x^2+2axy+ay^2+2x+4y = 0$;

(b) $x^2+2axy+(a+1)y^2+2x+4y = 0$.

17. Describe the 'circle' $x^2+y^2 = 0$.

## 1.18. The equation of the tangent at a point of a circle

If the coordinates of a point $P$ of a given circle are known, the gradient of the radius through $P$ may be found. Since the tangent at $P$ is perpendicular to this radius, the gradient of the tangent is then easily calculated.

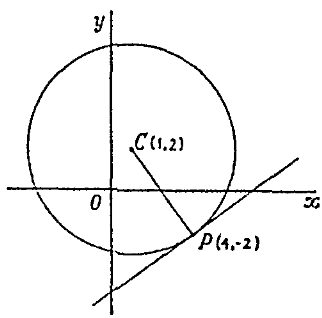


FIG. 1.24

EXAMPLE. Find the equation of the tangent at $P \equiv (4, -2)$ to the circle $(x-1)^2+(y-2)^2 = 25$.

*Solution.* Let the centre of the circle be $C \equiv (1, 2)$. The gradient of $CP$ is $\dfrac{-2-2}{4-1} = -\frac{4}{3}$. Hence the gradient of the tangent at $P$ is $\frac{3}{4}$.

Its equation is therefore

$$4y-3x = 4\times(-2)-3\times 4$$
$$= -20,$$

or

$$4y-3x+20 = 0.$$

## Exercise 1 T

1-6 Find the equation of the tangent at the given point to the given circle

1 $x^2+y^2 = 5$ $(1, 2)$

2 $x^2+y^2 = 20$ $(2, -4)$

3 $(x-2)^2+y^2 = 4$ $(0\ 4, 1\ 2)$

4 $(x-3)^2+(y+4)^2 = 40$ $(1, 2)$

5 $x^2+y^2-2x-2y = 0$ $(0, 0)$

6 $x^2+y^2+4x-8y = 30$ $(3, -1)$

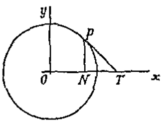
7 Find the equations of the tangents at $(-2, -1)$ and $(2, 3)$ to the circle $x^2+y^2+4x-6y-3 = 0$

8 Show that the line joining $A \equiv (2, 6)$ and $B \equiv (-6, 2)$ is a tangent to $x^2+y^2 = 20$ at the middle point of $AB$

9 Show that the tangent at $(3, 4)$ to $x^2+y^2 = 25$ is also the tangent at $(7, 1)$ to $x^2+y^2-2x+14y-50 = 0$

10 Find the equations of the tangents to $x^2+y^2 = 100$ at the points $A$ and $B$ where it meets $x = 8$ Find the coordinates of their point of intersection, $P$ Find the length of $PA$

11 Find the equations of the tangents at $(3, 4)$ and $(4, -3)$ to $x^2+y^2-14x-2y+25 = 0$ and write down the coordinates of their point of intersection



Ex 1 T 12

12 $P \equiv (x_1, y_1)$ is a point on the circle $x^2+y^2 = a^2$ Show that

$$x_1^2+y_1^2 = a^2$$

and the equation of the tangent at $P$ is $xx_1+yy_1 = a^2$ Find the intercepts made on the axes by this tangent

If the tangent at $P$ meets the $x$ axis at $T$ and $N$ is the foot of the perpendicular from $P$ to the $x$ axis, show that $ON\ OT = a^2$

## 1.19. Tangents to a curve

The tangent at a point $P$ of a circle may be drawn accurately by constructing it at right angles to the radius through $P$ No such construction exists for tangents to other curves

Consider for example $10y = x^2$ Draw this curve from the given table of values taking 1 in to represent 1 on each axis

| $x$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 1 | 0 | 0 1 | 0 4 | 0 9 | 1 6 | 2 5 | 3 6 | 4 9 |

Draw by eye the tangents at $x = 2$ and $x = 5$ Determine the

gradients of these tangents. Compare your answer with those obtained by other members of the class and note that all do not quite agree. This is rather unsatisfactory and in the next chapter we shall learn how to calculate the gradient of the tangent at any point of a curve whose equation is given. It will then be possible to show by calculation that the gradient of the tangent at $x = 2$ is exactly 0·4 and that at $x = 5$ it is 1. Compare your answers with these figures.

Although we cannot draw it accurately, we assume that at every point of the curve there is a definite tangent. Let

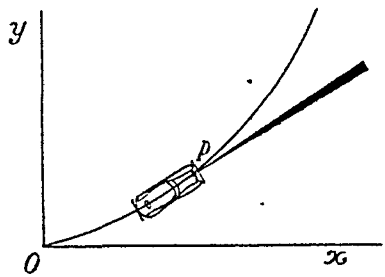



FIG. 1.25

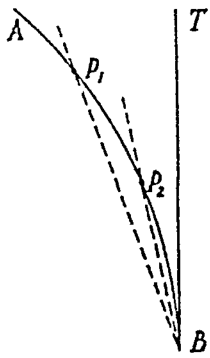



FIG. 1.26

$10y = x^2$ be the curve traced out by the headlamp of a car moving on a curved road and let $P$ denote any position of the lamp. Then if the headlamp throws a narrow beam of light straight ahead of the car, this beam would lie along part of the tangent at $P$ to the curve $10y = x^2$. The tangent maintains all along its length the instantaneous direction of motion of the lamp at $P$.

Now consider a more geometrical way of looking at the tangent at a point of a curve. $AB$ is a curved railway line on which a locomotive is travelling from $A$ to $B$. Imagine that it is night and that a boy crouches in a hole dug between the rails at $B$ with his eyes just above ground level. As the locomotive approaches he keeps a narrow horizontal beam of light focused on the bottom of the front coupling. The beam $BP$ passes through the positions $BP_1$, $BP_2$,... and settles down in the position $BT$ as the locomotive thunders over the boy's head. $BT$ is the tangent at $B$ to the curve $AB$.

$BP_1$, $BP_2$,... are chords (not to be regarded as terminating at $P_1$, $P_2$, etc.) of the curve $AB$. $BT$ is the position in which the variable chord $BP$ settles down as $P$ approaches $B$. The

tangent at $B$ is then said to be defined as the *limiting position* of the chord $BP$ as $P$ approaches $B$. The point $B$ is called the *point of contact* of the tangent.

### Exercise 1 U

1 Calculate the gradient of the chord joining (5, 2 5) and (6, 3 6) on $10y = x^2$

2 Calculate the gradient of the chord joining (5, 2 5) and (5 1, 2·601)

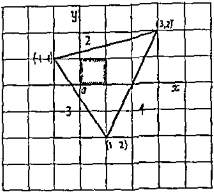



Fig 1 27

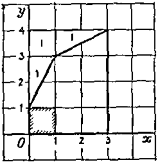



Fig 1 28

**1 20.** It is sometimes necessary to find the area of a polygon when the coordinates of its vertices are known. The method is best explained by an example.

Example 1 Find the area of the triangle whose vertices are $(-1, 1)$, $(3, 2)$, $(1, -2)$

*Solution* Fig 1 27 is a sketch of the triangle. Through the vertices of the triangle draw lines parallel to one or other of the axes so as to enclose the triangle in a rectangle. Then the area of the triangle is calculated by subtracting from the area of the rectangle the areas of three right angled triangles. The required area is $16-4-2-3 = 7$ sq units. The sq unit is the area of a square of side 1 e g the square on $(0, 0)$, $(1, 1)$ as opposite vertices. In questions in which dimensions are given, the area is first found in terms of sq units, and the area of 1 unit square is then found in terms of the scales of the axes.

Example 2 Fig 1 28 is the plan of a field on a scale of $\frac{1}{4}$ in (side of 1 square) $= 100$ yd. Find the area of the field.

*Solution* The area $= 12-1-1-1 = 9$ sq units. Each unit square represents $10^4$ sq yd. Hence the area of the field is $9 \times 10^4$ sq yd

## Exercise 1.V

1–3. Find the area of the triangle whose vertices are:

1. $(-1, 0)$, $(4, 0)$, $(1, 2)$.  2. $(-2, 1)$, $(2, 3)$, $(3, -1)$.

3. $(-2, -1)$, $(3, 2)$, $(1, 0)$.

4. Find the area of the quadrilateral whose vertices are $(1, 2)$, $(4, 0)$, $(2, -3)$, $(-1, -1)$. Show that the quadrilateral is a square.

5–6. The corners of a field bounded by straight hedges are given on a plan to a given scale by the coordinates given below. Find the area of the field.

5. $(1, 3)$, $(6, -1)$, $(1, -3)$, $(-3, 2)$. Scale: 1 unit on the axes = 50 yd.

6. $(-2\frac{1}{2}, 1)$, $(3, 3)$, $(5, -2)$, $(-1, -2)$. Scale: 3 units = 100 yd. (Answer in acres.)

7. The cross-section of a haystack is given by the straight lines joining the points $(-3, 0)$, $(-4, 6)$, $(0, 10)$, $(4, 6)$, $(3, 0)$ where 1 unit = 2 ft. Find the area of the cross-section.

## Miscellaneous Exercise 1.X

*Based on Sections 1·1–1·12.*

1. Find the equation of the line through $(3, -2)$ of gradient 2.

2. Find the equation of the line through $(5, -7)$ with direction angle 90°.

3. Find the direction angle of the line $3{\cdot}2x+2y+1 = 0$.

4. Write down the equation of the family of lines which make equal intercepts (with the same sign) on the axes. What is the direction angle of any one of these lines?

5. Which pair of the three following lines is parallel?
(a) $x-2y = 1$;  (b) $x+2y-4 = 0$;
(c) $y-x/2 = 1$.

6. Show that $(1, 5)$ is on each of the lines
(a) $y = x+4$;  (b) $3y+2x = 17$;
(c) $4y-5x = 15$.
Which is the steepest line?

7. Which of the points $(5, -2)$, $(-1, -1)$ is nearer $(3, 4)$?

8. $A \equiv (3, -4)$, $B \equiv (7, 2)$. $P$ is the middle point of $AB$, and $Q$ is the middle point of $AP$. Find the coordinates of $P$ and $Q$.

9. Find the equation of the line joining the middle point of $(-2, -7)$, $(4, 8)$ to the origin.

10. Show that the joins of $(-2, -4)$ to $(-5, 0)$ and $(6, 2)$ are perpendicular.

11 Find the equation of the line through $(4, -3)$ of gradient $-3$

12 Find the equation of the line through $(3, 1)$ parallel to $y = 2x+1$

13 Find the equation of the line through the intersection of $x+y = 1$, $x-y = 3$ parallel to $3y+5x = 2$

14 Describe the directions of the lines (b), (c) in relation to (a)
(a) $x+2y = 1$, (b) $2x-y = 2$, (c) $2x+4y = 5$

15 Find the equations of the lines through $(1, 1)$ inclined to the $x$ axis at $45°$

16 Find the equations of the diagonals of the rectangle whose sides are $x = 0$, $x = 10$, $y = 0$, $y = 12$ Find the coordinates of their point of intersection

17 Find the intercepts on the axes made by $10x-9y+36 = 0$

18 Find the angle between the lines $y+2x = 0$, $y+4x = 0$ to the nearest half degree

19 Find the equations of the lines through $(-2, 1)$ parallel and perpendicular to $x+5y+3 = 0$

20 A telephone wire leading in to a house is attached to the pole at a height of 25 ft and to the house at a height of 12 ft Midway between the pole and the house it sags 1 ft below the straight line joining its ends Find its height at this point

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21 Find the equation of the line through $(7, 1)$ of gradient $\frac{1}{2}$

22 Find the equation of the line through $(2, 1)$ parallel to $y = 0$

23 Find the equation of the perpendicular from the origin to $3y+4x = 25$ Find the coordinates of its point of intersection with the line

24 Describe the directions of (b) and (c) in relation to (a)
(a) $2x+3y = 2$, (b) $x/3+y/2 = 1$, (c) $3x = 2y+1$

25 If $\tan\alpha = 0\,6$ find the equations of the lines through $(5, 3)$ with direction angles $\alpha°$ and $(180-\alpha)°$

26 Find the equation of the line of the family $y = mx+1$ which is perpendicular to the line for which $m = 0\,8$

27 Write down the gradient of
(a) the join of $(-1, -2)$ to $(3, 6)$, (b) the line $6y-9x+5 = 0$

28 Which of the points $(1, 2)$, $(-1, 1)$, $(-4, -1)$, $(2\frac{1}{3}, 0)$, $(3, 3)$ are on $3x-5y+7 = 0$ ?

29 Find the middle point of the line joining $(0, -8)$ and $(0, 4)$ Find the equation of the perpendicular bisector of the line joining them

30 Which is nearer the origin, the point $(10, 9)$ or the point on $x+2y = 0$ whose $x$ coordinate is 12 ?

*Based on the whole of Chapter I*

31. Find the equation of the line through $(-10, 2)$ of gradient $-\frac{3}{4}$.

32. Find the equation of the perpendicular from $(1, 2)$ to $2y-x = 17$.

33. Find the equation of the perpendicular bisector of $(-1, 3)$, $(5, -1)$.

34. If $O$ is the origin and $P$ is any point on $y = 16$ write down the equation of the locus of the middle point of $OP$.

35. Find the centre and radius of the circle $x^2+y^2+2x-6y-6=0$.

36. Find the line of the family $y = mx+2$ which passes through $(3, -4)$.

37. $x+ay+a^2 = 0$ is the equation of a family of lines. What value of $a$ gives the $y$-axis ?

38. Show that the points $(1, 0)$, $(0, -1)$, $(200, 199)$ are in line and find the equation of the line.

39. Show that the lines $x/5+y/15 = 1$ and $2x-6y = 1$ are perpendicular.

40. Find the equation of the tangent at $(5, 2)$ to the circle $x^2+y^2-7x+2y+2 = 0$.

---

41. Find the equation of the lines through $(-4, 2)$ parallel and perpendicular to $3y+2x = 2$.

42. Find the equation of the line through $(1, 5)$ making the same angles with the axes as $x+2y = 1$ but not parallel to it.

43. A circle, centre the origin, passes through $(7, 1)$. What is its equation ?

44. A rectangle is drawn with one vertex at the origin, two sides along the axes, and area 100 sq. units. Find the equation of the locus of the vertex opposite to the origin.

45. Find the equations of the lines through $(7, 5)$: (*a*) parallel and perpendicular to $x+1 = 0$; (*b*) parallel and perpendicular to $x+5y+3 = 0$.

46. Find the angles between the following pairs of lines to the nearest degree.

(*a*) $2x-y+7 = 0$, $3x-6y+2 = 0$;

(*b*) $x+3y+1 = 0$, $3x-y+2 = 0$;

(*c*) $2x+7y+1 = 0$, $4x+14y+9 = 0$.

238

47. $y+x = 0$, $y-2x = 6$, $y+x = 3$, $y-2x = 0$ are the sides of a quadrilateral. Show that it is a parallelogram, and find the coordinates of its vertices.

48. For what value of $a$ does $2x^2+ay^2+2ax+4y-14 = 0$ represent a circle ? Find the centre and radius of this circle.

B6 : 23 H8·1

49 Find the area of the triangle $(1, 2)$, $(-1, -1)$ $(2, -3)$

50 Find the coordinates of the foot of the perpendicular from the origin to $4x+y = 8\frac{1}{2}$

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51 Show that $(5, 5)$, $(\frac{4}{5}, -\frac{2}{5})$, $(-1, 1)$ lie on a circle with centre $(2, 3)$ which also passes through the origin

52 What is the value of $k$ if $3y = 7x+k$ (i) passes through the origin (ii) makes an intercept of 2 on the $x$ axis?

53 Write down the equation of the locus of a point whose distance from the origin is 6 Write down the equations of the two straight lines which form the locus of a point whose perpendicular distance from $y+4 = 0$ is 6

54 Show that $2x+5y = 22$ passes through the middle point of the line joining $(3, 1)$ to $(4, 5)$

55 For what value of $a$ does the equation $x^2+2axy+y^2+2x = 0$ represent a circle? Find the centre and radius of this circle

56 Find the equation of the perpendicular bisector of the join of $(1, 2)$ and $(5, 8)$

57 Find the gradients of the sides of the quadrilateral $(1, 2)$, $(2, -1)$, $(0, -2)$, $(-1, 1)$ Verify that it is a parallelogram

58 A point moves so that its distance from $(0, 3)$ is twice its distance from the origin Show that the equation of its locus is a circle and find its radius and centre

59 Show, in the easiest way you can, that $(1, 2)$ is a point of intersection of $2x-3y+4 = 0$ and $x^2+y^2+4x+8y = 25$

60 Find the equation of the reflection of $y-2x = 5$ in the $x$ axis

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61 Find the equation of the tangent at $(2, -4)$ to

$$x^2+y^2-4x+2y = 4$$

62 Find the equation of the circle through $(0, 0)$, $(0, 2)$, $(1, 0)$ Find its centre and radius

63 A point moves so that the square of its distance from $(-1, 0)$ is 4 units greater than the square of its distance from $(3, 0)$ Find the equation of its locus

64 Find the equation of the circle having the same centre as $x^2+y^2+2x-6y+6 = 0$ *and radius twice as great*

65 Find the equation of the line joining $(1, 2)$ and $(5, 7)$ Also find the equation of the line joining $(7, 6)$ and $(-1, 3)$ and show that the lines bisect each other

66 Find the equation of the line joining $(1, -2)$, $(7, -2)$ Find the equation of its perpendicular bisector

67 Find the area of the octagon $(0, 0)$, $(0, 1)$, $(1, 2)$, $(2, 2)$, $(3, 1)$, $(3, 0)$, $(2, -1)$, $(1, -1)$

68. $P \equiv (-3, 2)$, the gradient of $PQ = \frac{3}{4}$, and the length of $PQ$ is 10. Find the coordinates of $Q$ (2 answers).

69. Find the equation of the circle with centre $(1, -2)$ passing through $(4, 2)$. What is its radius?

70. Find the equation of the reflection of the line joining $(1, -1)$ and $(-2, 3\frac{1}{2})$ in the $y$-axis.

## Miscellaneous Exercise 1.Y

1. Find the equation of a line of twice the gradient of $3x-2y = 4$ passing through $(1, 3)$. Find the coordinates of the point of intersection of the two lines.

2. Find the equation of the line joining $(1\frac{1}{2}, -2\frac{1}{2})$ to the origin. Find the equation of the perpendicular to this line passing through $(0, 4\frac{1}{5})$.

3. Find the equations of the reflections in the axes of $x+2y = 1$.

4. $A \equiv (-1, -1)$, $B \equiv (3, 1)$, $C \equiv (1, 3)$. $D, E, F$ are the middle points of $BC, CA, AB$. Find the equations of the sides of the triangle $DEF$.

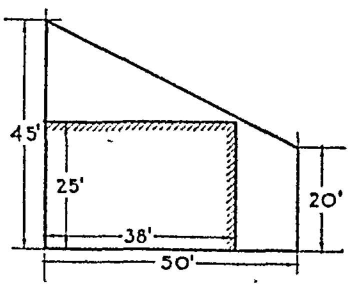
5. Find, to the nearest $\frac{1}{2}$ degree, the angles of the triangle whose sides are $y+2x = 0$, $2y-x+3 = 0$, $x = 0$.

6. Three of the following points are on the same straight line. Find the equation of the line. $(-9, 4)$, $(-1, -1)$, $(-4, 1)$, $(3, -4)$, $(6, -6)$.

7. $OAB$ is a triangle, right-angled at $A$. $O$ is the origin and the gradients of $OA$, $OB$ are $2, \frac{1}{3}$. The point $(2, 1\frac{1}{2})$ is on $AB$. Find the equations of the three sides and the coordinates of $A$ and $B$.

8. A line is drawn through $(3, 4)$ on the circle $x^2+y^2 = 25$ parallel to $2y-x = 0$. Find the coordinates of the point in which it meets the circle again. Find the coordinates of the point of intersection of the tangent at this point and the tangent at $(3, 4)$.

9. An aerial is attached to a chimney at a height of 45 ft. above the ground. Its other end is supported by a pole 20 ft. high, 50 ft. from the base of the chimney. Find the vertical height at which the wire passes over the corner $A$ of the outbuilding shown in the diagram.




10. Two vertical walls are parallel and at a horizontal distance of 30 yd. apart. A wire is stretched from the foot of one wall to a window in the other wall directly opposite and 20 yd. above the ground. A second wire is stretched from a window 40 yd. above the

ground in the first wall to the foot of the second Both wires are in a plane at right angles to the walls so that they cross Find the height of their intersection above the ground

11 Find the centre and radius of the circle

$$3x^2+3y^2-4x+6y+3=0$$

Find the equation of the tangent at the point (1 2, —0 6)

12 $A \equiv (0, 0), B \equiv (-2, 1), C \equiv (-3, -1)$ Find the equations of the perpendicular bisectors of $AB$, $BC$ and the coordinates of their point of intersection, $O$ Find the distance of $O$ from $A$, $B$, $C$

13 Find the locus of a point which is equidistant from $A \equiv (-1, 3)$ and $B \equiv (5, -1)$ Which of the following points are equidistant from $A$ and $B$? $P \equiv (3, 2)$, $Q \equiv (4, 4)$, $R \equiv (-2, -5)$

14 $O$ is the origin and $A$ is $(4, 0)$ Find the locus of $P$ if $OP^2-PA^2 = 1$

15 $A \equiv (-2, 0), B \equiv (1, 0)$ $P$ is a point such that $AP = 2PB$ Show that the locus of $P$ is a circle passing through the origin Show that the centre of the circle is on the $x$ axis and find its radius

16 The opposite ends of the diameter of a circle are (1, 1), (4, 5) Find the coordinates of the centre and the radius Find the equation of the circle

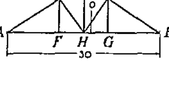
17 Show that (—1, 1), (1, 2), (3, —2), (1, —3) are the vertices of a rectangle Show that the sides are in the ratio 2 1 and find the length of the diagonals

18 A pair of opposite vertices of a parallelogram are (1, 2), (—5 6) What are the coordinates of the intersection of the diagonals? If another vertex is (1, —3), what are the coordinates of the fourth vertex?

19 $ABC$ is a triangle and the middle points of $BC$, $CA$, $AB$ are $D$, $E$, $F$ What can you say about $BC$ and $EF$? If the coordinates of $D$, $E$, $F$ are (2, 0), (1, 1) (2, 2), find the equations of $BC$, $CA$, $AB$ and the coordinates of $A$, $B$, $C$

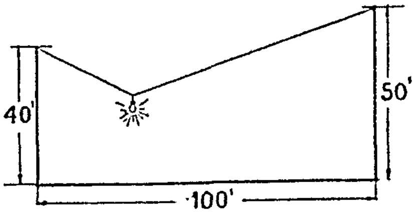
20 Show that the perpendicular bisector of the line joining (0, 1) to (—1, 2) passes through the middle point of the line joining (0, —1) to (2, 3)

21 Find the area of the quadrilateral whose vertices are (0, 0), (—2, 1), (1, 6), (5, 2)

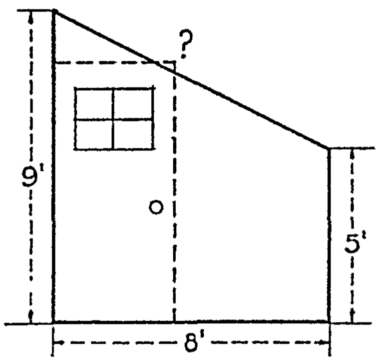



22 The figure shows a symmetrical framework of girders for supporting a roof $HE$, $HD$ are perpendicular to $BC$, $AC$ $EG$, $DF$ are vertical $HC = 10$ ft $AB = 30$ ft Choose your own axes and find the length of $EG$ and $DF$

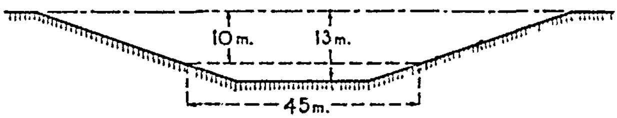
23. A lamp is suspended from the buildings on opposite sides of a street by wires sloping at $\alpha$, $\beta$ to the horizontal, where $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$. The wires are in a vertical plane at right angles to the fronts of the buildings. Choose your own axes and find the height of the lamp above the roadway and its horizontal distance from the nearer wall.




24. A man has a lean-to greenhouse of the dimensions shown in the sketch. He buys a second-hand door 3 ft. 6 in. by 7 ft. 6 in. which he wants to fit at one end of the greenhouse. Is he a mug or not?




25. The sketch shows the cross-section of the Suez Canal. The sides rise 1 (vertically) in 3 (horizontally). The canal is 13 metres




deep and is 45 metres wide at a depth of 10 metres. Find the width of the canal at the bottom and surface and the area of the cross-section.

26. If $A \equiv (-1\frac{1}{2}, 2\frac{1}{2})$, $B \equiv (-1, 1)$, $C \equiv (3, 4)$, prove that the triangle $ABC$ is right-angled and find at which vertex the right angle occurs. Find the equation of the circle passing through $A$, $B$, $C$.

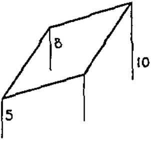
27. Find the equation of the tangent at $P \equiv (4, 3)$ to

$$x^2+y^2-6x-2y+5 = 0.$$

Find the coordinates of the points $A$ and $B$ in which this tangent meets the circle $x^2+y^2-6x-2y = 0$. Show that $P$ is the middle point of $AB$.

28. The equations of the sides of a triangle are $5x-3y+4 = 0$, $4x+y-7 = 0$, $3x+5y+16 = 0$. Find the coordinates of the vertices and show that the triangle is isosceles.

29 A square board is held at an angle to the horizontal floor with three of its corners at heights of 5, 8, 10 ft, as shown in the sketch Find (i) the height of the centre of the plate; (ii) the height of the fourth corner

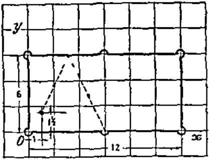



30 $A \equiv (-2, 1)$, $B \equiv (-2, -3)$, $C \equiv (4, -2)$, $D \equiv (1\frac{3}{4}, 2\frac{1}{4})$ Find the equations of the perpendicular bisectors of $AB$ and $BC$ Find the coordinates of $K$, the point in which these bisectors meet Find $KA^2$, $KB^2$, $KC^2$, $KD^2$ Is the quadrilateral $ABCD$ cyclic?

31 $A \equiv (-2, 2)$, $B \equiv (-3, -2)$, $C \equiv (3, 0)$, $D \equiv (4, 4)$ Verify that $ABCD$ is a parallelogram and that

$$AB^2+BC^2+CD^2+DA^2 = AC^2+BD^2$$

32 A billiard table is 6 ft by 12 ft It is desired to strike a ball $1\frac{1}{2}$ ft and 1 ft from two adjacent sides (as shown in the diagram) so that it falls into the middle pocket after one rebound Use the axes marked on the figure and denote the gradient of the first part of the path by $m$ Find $m$ and find the point on the cushion at which the ball should be aimed




33 $A \equiv (-2, 4)$, $B \equiv (-2, -2)$, $C \equiv (4, 2)$ Find the equations of the perpendiculars through $B$ and $C$ to the opposite sides of triangle $ABC$ Find the coordinates of their point of intersection, $H$ Verify that $HA$ is at right angles to $BC$

34 $A \equiv (1, 5)$, $B \equiv (-4, 0)$, $C \equiv (4, 0)$ Find the equations of the medians of the triangle $ABC$ through $A$ and $B$ Find the coordinates of their point of intersection Find the equation of the line joining this point of intersection to the middle point of $AB$ and verify that it passes through $C$

35 Measurements of the mean lengths of the facial and cranial regions of the skulls of 30 sheep dogs at different ages have been made Their mean values at each age are given in the following table †

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Facial region ($x$) in mm | 22 0 | 48 3 | 58 0 | 73 5 | 89 1 | 102 0 | 112 0 |
| Cranial region ($y$) in mm | 42 0 | 65 3 | 74 5 | 85 5 | 99 3 | 112 6 | 120 0 |

† From D Arcy Thompson, *Growth and Form*, p 210

Show that a linear law connects $y$ and $x$ and by drawing the best straight line you can, find this law.

36. Find the area of the triangle formed by the line $3x+7y = 42$ and the axes. Find the equation of the line through (12, 0) which has an equal area between itself and the axes.

37. Show that the distance from (4, 0) to the point $(h, 4\sqrt{h})$ is $4+h$.

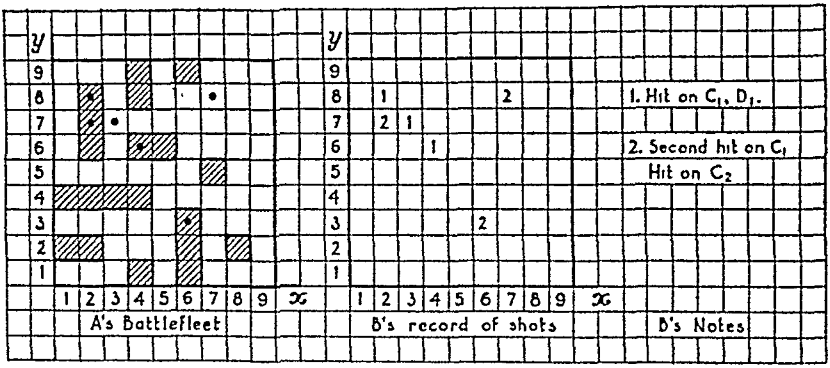
38. The efficiency, $E$, of a petrol engine for various values of the compression ratio, $r$, is given in the following table:

| | | | | | | |
|---|---|---|---|---|---|---|
| $E$ | 0·414 | 0·426 | 0·442 | 0·452 | 0·466 | 0·475 |
| $r$ | 3·8 | 4·0 | 4·3 | 4·5 | 4·8 | 5·0 |

Draw a graph and show that the law connecting $E$ and $r$ is approximately linear. Find the best law you can to fit the figures.

## Miscellaneous Exercise 1.Z

1. Practice in giving positions by coordinates is afforded by the game of Battleships.




The two players, $A$ and $B$, each draw two sets of $9\times 9$ small squares to form two large squares. On one square each places his battle fleet according to his own idea of tactics and conceals their disposition from his opponent. Each battle fleet consists of 1 battleship (4 squares in line), 2 cruisers (3 squares in line), 3 destroyers (2 squares in line), 4 submarines (1 square). The positions of these vessels, indicated by shading, must obey the following laws:

(i) No two vessels may touch, even at a corner.

(ii) Any vessel may have contact with the boundary of the large square but not over more than the side of one small square. (It follows that the corner squares cannot be occupied.)

The game is played by each player in turn firing salvoes of three shots. The first to sink all his opponent's ships wins. Suppose $A$ disposes his fleet as shown in the diagram and $B$ opens fire first. He

announces his salvo by giving the coordinate of the top right hand corners of the squares in which his shots fall with reference to $x$ and $y$ axes laid down in the usual way along two sides of the large square $A$ records the salvo by dots on his battle fleet diagram and announces the hits without giving away which shots did the damage $B$ puts the figure 1 in the three corresponding squares of his record diagram and makes notes of the hits Thus the opening stages of the game illustrated above would be

$B$ (2, 8), (3, 7), (4, 6)
$A$ One hit on cruiser One hit on destroyer

Then $A$ fires his salvo and $B$ announces the hits $B$ then continues

$B$ (2, 7), (6, 3), (7, 8)
$A$ Second hit on cruiser Hit on the other cruiser

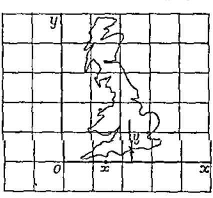
A vessel is sunk when all the squares occupied by it are hit As the position of each of the opponent's vessels is determined it will be found useful to mark it on the record diagram since by law 1 it is useless to fire at any of the adjoining squares

Great care is necessary in giving accurately the coordinates of the shots otherwise the game is ruined In particular, the order $x$ coordinate before $y$ coordinate must be rigidly adhered to

2 *The National Grid*

On the National Grid, the position of any point in England and Wales is defined by the east and north steps by which the point may be reached from an origin which is situated to the south west of Land's End The lengths of the steps are given on the metric system, in metres if great accuracy is desired, but more usually to the nearest 100 metres (= 0 1 km )

If coordinate axes are drawn through the origin, so that the positive $x$ axis has the direction east and the positive $y$ axis has the direction north, the east and north steps by which a given position is reached are the $x$ and $y$ coordinates of its position with respect to these axes Hence the National Grid is a method of defining position by means of coordinates On a map like the One Inch Ordnance Survey Map of England and Wales which, for portability, is divided into sheets,




it is impossible to refer directly to the origin Hence the families of lines $x = k_1, y = k_2$ respectively parallel to the north–south and east–west lines at the origin, are printed on the map at intervals of 1 km These lines serve the same purpose as the lines on a sheet of

graph paper. They cover the map with a network of squares, each of side 1 km., which form the *grid*. The distance of each line east or north of the origin is printed on the margin of the map. Thus on the map on page 61, which is a reproduction of a portion of Sheet 162 of the New Popular Edition, the right-hand north–south line of the map has the figures 610 printed at its northern end. These figures denote that the line is 610 km. east of the origin. The hundreds figure is only printed on each tenth line. Similarly we can find on the map the line of the family $y = k_2$ which is 210 km. north of the origin. The figure 2 is omitted on the remaining lines of this family shown on the portion of the map which is reproduced here.

Now consider the eastern point of Mersea Island, called Mersea Stone. This point is a little to the east and north of the intersection of the lines $x = (6)07$, $y = (2)15$. By estimation it is $\frac{3}{10}$ km. east and $\frac{3}{10}$ km. north of this intersection. Its position with reference to the origin can be defined as $x = (6)073$, $y = (2)153$, where unit $x$ and unit $y$ is $\frac{1}{10}$ km. The first figures are put in brackets because they are usually omitted. This is because the remaining figures define the point without ambiguity if we know the district in which the point lies or the map sheet that is being used.† The two coordinates are now put together, the $x$-coordinate being placed before the $y$-coordinate. Thus 073153 is the *Normal National Grid Reference* of Mersea Stone.‡

Now consider the converse procedure, the identification of a given reference, e.g. 082166. The $x$- and $y$-coordinates separately are 082 and 166. We first find the intersection of the lines marked 08 and 16. [Note that there is only one such intersection on a given sheet since every sheet is much smaller than a square of side 100 km.] The position to be identified, therefore, lies in the square of side 1 km. to the north and east of this intersection. Since the unit figures in the coordinates are 2 and 6 respectively we estimate the position reached from the SW corner of the square by the step $\frac{2}{10}$ km. E. followed by the step $\frac{6}{10}$ km. N. This position is *Brightlingsea Railway Station.*

2.1–3. Identify the positions defined by the references:

2.1. 095147. 2.2. 056188. 2.3. 098176.

2.4–6. Give the references of these places:

2.4. East Mersea Church. 2.5. St. Osyth Stone Point.

2.6. Westmarsh Point (Brightlingsea, junction of R. Colne and Brightlingsea Creek).

2.7. Give the references of the ends of Rat Island (west side of R. Colne).

† For example, the point $x = (5)073$, $y = (3)153$ is 100 km. west and 100 km. north of Mersea Stone.

‡ In cases where the use of the normal reference might be ambiguous, the hundreds figures of the $x$- and $y$-coordinates are given separately in this order. We then have the *Full National Grid Reference*. Thus the full reference for Mersea Stone would be 62/073153.

2 8 There is a nasty patch of sand off Mersea Island called the Coccum Hills If the reference is 066126, identify the patch on the map

2 9 Brightlingsea Parish Church is over a mile from the town Identify it from the reference 077187

The line passing through this church and a tower near Westmarsh Point touches the edge of the mud on the west side of Brightlingsea Reach Give the map reference of the tower What is the approximate bearing of this danger line from seaward ?

2 10 Give the reference of the position of a ship which has sailed 10 km south from the position 090120

2 11 There is a navigational buoy called the Inner Bench Head buoy in the position 087115 Plot this buoy

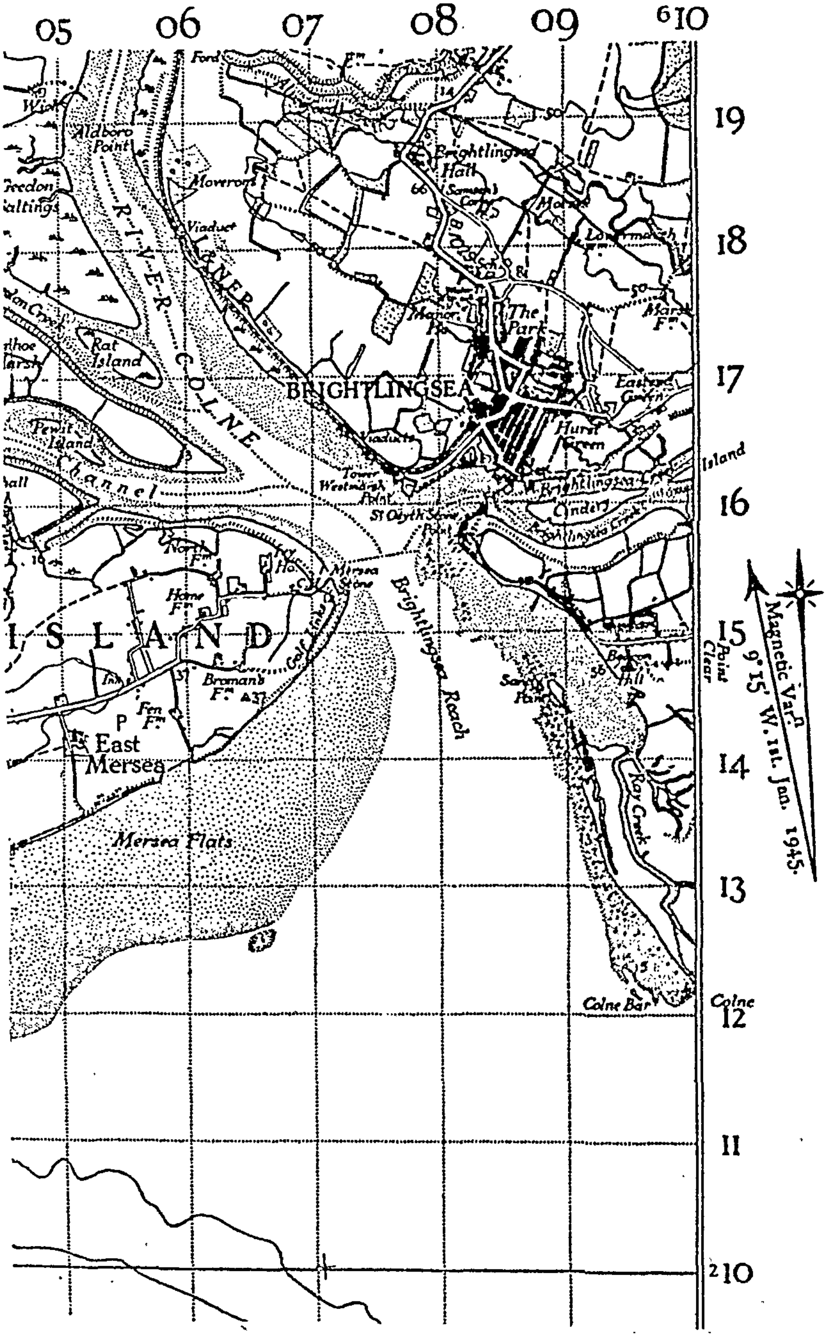
Find, approximately, by using a protractor, the course from the Inner Bench Head buoy to a point in Brightlingsea Reach midway between Mersea Stone and St Osyth Stone Point

[Note that the north direction, given on the map, is inclined to the grid lines at about 2° and allow for this This angle is the inclination of the direction north near Brightlingsea to the direction north at the origin of the grid ]

2 12 *Sailing directions for Brightlingsea* Identify the references given in the following directions intended for a stranger who is visiting Brightlingsea for the first time

There is safe anchorage for small yachts in the R Colne off Bateman's Tower, 076162, although a better anchorage in strong winds is at the entrance to Pyefleet Creek, 062161 Landing is possible between half flood and half ebb at a shingle beach near Bateman's Tower and water may be had from a standpipe close by At other times it is necessary to row up Brightlingsea Creek to the Town Hard, 087161, where it is possible to land at all states of the tide The Post Office is on the corner at 086169 and provisions may be obtained from several shops near by

3 The story goes that when Descartes went to Heaven, the mathematicians who were already there crowded round him and begged him to explain his new system of geometry about which the worldlings were so excited So he took 12 of the best mathematicians into a room and seated them at 12 desks as in the figure on p 62 Then he laid down his axes and scales as shown in the sketch and said 'Now, gentlemen, to convince you that my coordinates define points without ambiguity, for the remainder of this demonstration we will refer to each other not by our worldly names but by the coordinates of the centre of the desk which each of us occupies Tell me (0, 0), you were a Scot and should appreciate logic, do you agree ?'




University Press, Oxford

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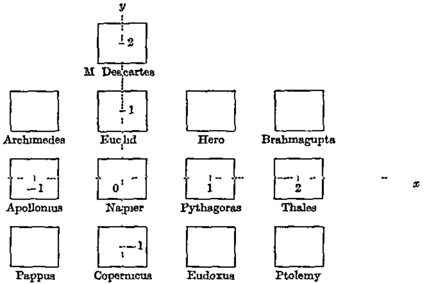
'No, I do not,' replied (0, 0) 'My name is famous among all the worldlings who use my logarithms and I cannot exchange it for two such paltry numbers, even for a moment' Before (0, 2) could reply, (−1, 1) spoke 'Nay, (0, 0), do not hinder the demonstration (−1, 0), (−1, −1) and I considered these questions many years ago and now we are most anxious to hear how coordinates can be applied to the problems of geometry Pray continue (0, 2)' But for some time (1, 0) had been showing signs of excitement 'By my Theorem,' he




burst out, 'the square of the distance from (−1, −1) to (2, 1) equals the sum of the squares of the distances from (−1, −1) to (2, −1) and from (2, −1) to (2, 1)' Here (2, 0) interrupted 'Yes, and you yourself, (1, 0), are at the vertex of an isosceles triangle whose base is formed by (0, 1) and (0, −1)' (0, 2)'s attention being thus directed to (0, 1) he asked him what he was writing on his desk 'Quod erat demonstrandum' was the somewhat embarrassed reply And taking the hint, (0, 2) led the way out of the room

Read the story through, replacing each pair of coordinates by a name

4 The vertices of a quadrilateral are (1, 3), (3, 0), (0, −2), (−2, 1) Find the gradients of the sides and diagonals and show that it is a square

5 Show that the family of straight lines, $y-2 = m(x-4)$ passes through (4, 2) for every value of $m$ Give a sketch for $m = 1$, 3, −2, 0

6 The centre of a square is (1, 0) and one vertex is $(2\frac{1}{2}, 2)$ Find the coordinates of the other vertices and the equations of the sides

7. (*a*) A rectangular box is propped against a wall so that four of its edges are horizontal. One is in contact with the ground, another is in contact with the wall 8 in. above the ground, and a third is 4 in. above the ground. Find the height of (i) the centre of the box, (ii) the highest horizontal edge.

(*b*) A rectangular box is held with one edge in contact with the ground, and must, therefore, have three other edges parallel to the ground. Show that the height of the highest edge equals the sum of the heights of the remaining two.

8. Show that the six points $(0, 5)$, $(-1, 2)$, $(-2, -1)$, $(2, 1)$, $(2\frac{2}{3}, -\frac{1}{3})$, $(5, 0)$ lie in threes on four straight lines. Find the equations of the straight lines.

9. Find the coordinates of a point which is equidistant from $(0, 3)$, $(7, 10)$, $(8, 7)$.

10. Show that a circle can be drawn through the points $(-1, 0)$, $(8, 0)$, $(0, -2)$, $(0, 4)$. Find its equation.

11. Show that if a circle touches both axes, the point of contact with the $x$-axis being $(a, 0)$, then it must touch the $y$-axis either at $(0, a)$ or $(0, -a)$. Find the equation of the circle in each case.

12. Show that (i) $x^2+y^2-4x-2y = 5$

and (ii) $x^2+y^2-4x-2y = 95$

are concentric circles and find the coordinates of the centre. Show that $A \equiv (-8, 1)$ and $B \equiv (10, 7)$ are on the circle (ii) and the middle point of $AB$ is on the circle (i).

13. If the lines $a_1x+b_1y+c_1 = 0$ and $a_2x+b_2y+c_2 = 0$ are perpendicular, show that $a_1a_2+b_1b_2 = 0$. Show that this condition is more general than the relation between gradients since it applies when the lines are $x = a$, $y = b$.

14. The equations of the sides of a triangle are $4y-3x = 13$, $y+4x = 8$, $5y+x = 2$. Find the coordinates of the vertices. Find the equations of the altitudes of the triangle which are perpendicular to $y+4x = 8$ and $5y+x = 2$. Find the coordinates of the intersection of these altitudes, and show that this point lies on the third altitude.

15. The axes can be chosen so that the vertices of any triangle are $(a, 0)$, $(b, 0)$, $(0, c)$. Draw the triangle and mark in the axes. Find the equations of the altitudes through $(a, 0)$, $(b, 0)$ and show that the $x$-coordinate of their point of intersection is 0. Show that this proves that the altitudes of any triangle meet at a point. (This point is called the *ortho-centre* of the triangle.)

16. A point moves so that its distance from $(1, 0)$ is $k$ times its distance from $(-1, 0)$. Find the equation of its locus and recognize it as a circle. Examine and explain the result when $k = 1$.

17 $B \equiv (3, 4)$ and is the middle point of $AC$

(i) If $C \equiv (5, 5)$, find the coordinates of $A$

(ii) If $C \equiv (x_1, y_1)$, find the coordinates of $A$

(iii) If $A$ is any point on $y+2x-5 = 0$, find the locus of $C$

18 A painter's ladder is set up against a vertical wall in a vertical plane at right angles to the wall so that its foot is on horizontal ground 8 ft out from the wall and its top 30 ft above the bottom of the wall The painter's shoulder is 5 ft above his feet As the painter ascends the ladder, find the equation of the locus of his shoulder referred to horizontal and vertical axes through the foot of the wall (Assume the painter remains vertical all the time)

If he can paint comfortably down to the level of his shoulder as long as it is not more than $2\frac{1}{2}$ ft out from the wall, down to what height can he paint without moving the ladder?

19 Any point $(x_1, y_1)$ is taken and a line is drawn through it cutting the $x$ axis at $A$ and the $y$ axis at $B$ so that $(x_1, y_1)$ is the middle point of $AB$

(i) Find the equation of the line

(ii) Find the locus of $(x_1, y_1)$ if it is chosen so that the area of the triangle $OAB = a^2$, where $O$ is the origin

(iii) Find the locus of $(x_1, y_1)$ if it is chosen so that the length of $AB$ always equals $b$

20 $A \equiv (-2, -1)$, $B \equiv (4, 1)$, $C \equiv (1, 3)$ $D$ and $F$ are the middle points of $BC$, $AB$ Find the equations of $AD$, $CF$ and the coordinates of $G$, their point of intersection Find the coordinates of $X$ where $G$ is the middle point of $BX$ Find the equations of the lines through $X$ parallel to $AG$ and $CG$, and show that they pass through $C$ and $A$ respectively

21 Show that the four points $(a, 0)$, $(a+b, a)$, $(b, a+b)$, $(0, b)$ form the corners of a square ($a, b$ may be taken to be positive The simplest method is to draw a sketch and use congruent triangles) Find the coordinates of the centre, $C$, of the square If a square piece of cardboard is fitted exactly to the figure and then moved so that the vertices originally at $(a, 0)$, $(0, b)$ remain on their respective axes, show that $C$ moves on part of a straight line and find its equation

22 Find the distance between $(h, k)$ and $(h+8, k+15)$ If $(h+8, k+15)$ moves on the line $y = 3x-7$, what is the locus of $(h, k)$?

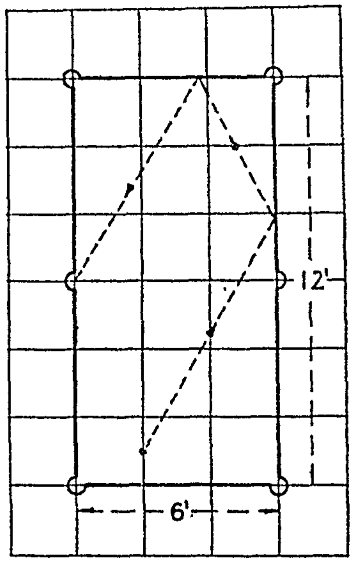
23 (a) The line $3x+2y = 48$ cuts the $x$ axis at $A$ and the $y$ axis at $B$ Find the area of the triangle $OAB$, where $O$ is the origin

(b) The equation of a family of lines is $x+a^2y = 2a$, where $a$ can be given any value Using the notation of (a) find $OA$ $OB$

[This result can be used to draw quickly a number of lines of the family. Do this, taking the lines fairly close together, and you will be surprised to see how they outline a smoother curve than you could possibly draw by joining points. A curve obtained in this way is called the *envelope* of the family.]

24. The billiard ball shown in the sketch is 2 ft. and 1 ft. from adjacent edges and is to be sent into the middle pocket after two rebounds.

Find the point on the first cushion to be struck at which the ball should be aimed. [Hint: Let the gradient of the first leg of the path be $m$.]




25. $A \equiv (0, 0)$, $B \equiv (0, b)$, $C \equiv (a, b)$, $D \equiv (a, 0)$. If $M$ is the middle point of $BC$ and $DM$ meets $AC$ at $K$, show that $AK = 2KC$.

26. $O$ is the origin and $A \equiv (-4, 0)$, $B \equiv (2, 0)$, $C \equiv (0, 6)$. $M$ is the middle point of $BC$. $AM$ meets the $y$-axis at $D$. $BD$ meets $AC$ at $N$. Show that $ON$ is parallel to $BC$.

27. $A \equiv (0, a)$, $B \equiv (0, -a)$, $P \equiv (b, 0)$, $Q \equiv (-c, 0)$, ($a$, $b$, $c$ positive and $b \neq c$). $AQ$, $BP$ meet at $H$. $AP$, $BQ$ meet at $K$. Prove that $HK$ is parallel to $AB$.

28. Find the coordinates of the two points on $7y+12 = x$ whose distance from $(1, 2)$ is 5.

29. In 1937 a firm was building yachts of ($a$) 12 tons for £720, ($b$) 4 tons for £199, ($c$) $2\frac{1}{2}$ tons for £100. Draw a graph and show that the law connecting tonnage and price is very nearly linear. Find it approximately from the graph. The firm also built a 9-ton yacht for £575. Does this fit the law?

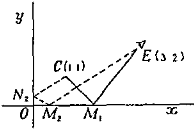
30. In 1936 one firm quoted marine engines as follows: 6 h.p., £65; 12 h.p., £92; 20 h.p., £130. Another firm quoted 7 h.p., £71; 10 h.p., £82; 16 h.p., £105. Show that the law connecting h.p. and price is approximately linear in each case and find the best law you can from graphs. What would be the h.p. of an engine which both firms sold at the same price?

31. The breaking strain of manila rope of various sizes and its weight are compared in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Breaking strain (cwt.) | 6 | 9 | 11 | 16 | 22 | 28 | 40 | 60 |
| Weight of 100 fathoms (lb.) | 18 | 22 | 28 | 45 | 60 | 80 | 125 | 170 |

Show that the law connecting them is approximately linear and find the best linear law you can from the graph

32 The figure shows two mirrors $Ox$, $Oy$ at right angles The reflection of a candle flame at (1, 1) is viewed by a man whose eye is at (3, 2) Find the equations of $CM_1$ and $M_1E$ which form the path of the light ray

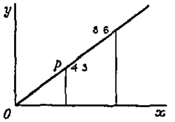



We may now consider whether the man can see a second reflection of the candle produced by a light ray following the track $CN_2M_2E$ He can if $N_2$ has a positive $y$ coordinate Find the equations of $CN_2$, $N_2M_2$, $M_2E$ Find the equation of the boundary separating the part of the first quadrant in which a second image is visible, from the part in which it is not visible

## 33 *Lattices*

33 1 The families of lines $x = n$, $y = k$, where $n$ and $k$ are positive or negative integers, cover the plane with a network of unit squares and are said to form a *lattice* To construct such a lattice take a sheet of graph paper, choose an origin, and number each set of parallel lines on the paper $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$

The intersections of the lines are points which may be used to represent fractions The fraction $\frac{3}{4}$ can be plotted as the point $P \equiv (4, 3)$ Then the value of the fraction is given by the gradient of the line joining $P$ to the origin $O$




The fraction $\frac{6}{8}$ $(= \frac{3}{4})$ is represented by the point (8, 6) which is also on the line $OP$

Construct, on a large sheet of graph paper, a line through $O$ of gradient $\sqrt{2}$ This may be done by drawing a large square, of side 100 say, and setting off the diagonal along the line $x = 100$ so that it stretches from $y = 0$ to $y = 100\sqrt{2}$ Join the origin to $(100, 100\sqrt{2})$ by a length of black cotton stretched tight Then points of the lattice which lie near the thread give fractions whose value is nearly $\sqrt{2}$ In this way I found that $\frac{99}{70}$ and $\frac{140}{99}$ are approximations to $\sqrt{2}$ Check these and find other approximations If you evaluate these fractions to 4 decimal figures and compare with the tabulated value of $\sqrt{2}$ you will see how good the approximations are

33.2. *Farey Series*

Draw on a sheet of graph paper the triangle whose sides are $y = x$, $x = 4$, $y = 0$. Consider the points of the lattice $x = n$, $y = k$ ($n$, $k$ integers) which lie within or on the sides of this triangle. Note the following facts about the fractions represented by these points:

(1) each fraction is not greater than 1;

(2) the denominator of each fraction is not greater than 4.

Now place a sheet of paper so that two edges lie along the positive $x$- and $y$-axes. Slowly rotate the paper counterclockwise about the corner at the origin and as each point of the lattice is uncovered write down the fraction which it represents. Sometimes the paper uncovers two or more points simultaneously. Then only write down the fraction corresponding to the point nearest the origin. (The fractions represented by the other points cancel down to this one.) When the paper has rotated slightly the points (1, 0), (2, 0), (3, 0), (4, 0) are uncovered. Write down $\frac{0}{1}$ corresponding to the point nearest the origin. The next point to appear is (4, 1). Write down $\frac{1}{4}$. The complete set of fractions in the order in which they appear will be found to be $\frac{0}{1}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{1}$. This series contains all the positive fractions (in their lowest terms), not greater than 1 and with denominators not exceeding 4, *arranged in increasing order of magnitude.* (This follows from the way in which the gradient of the edge of the paper changes.) It is called the *Farey Series of order 4.*

The series has two remarkable properties:

(1) If $h_1/k_1$, $h_2/k_2$ are two consecutive fractions,

$$k_1 h_2 - h_1 k_2 = 1.$$

(2) If $h_1/k_1$, $h_2/k_2$, $h_3/k_3$ are three consecutive fractions,

$$\frac{h_1+h_3}{k_1+k_3} \text{ reduced to its lowest terms} = \frac{h_2}{k_2}.$$

Test the series for these properties.

33.21. Obtain the Farey Series of orders 5 and 7 and test for these properties.

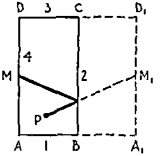
33.22. Determine graphically which of the fractions $\frac{9}{13}$, $\frac{5}{7}$ is the larger and check by arithmetic.

33.23. Arrange the following fractions in order of increasing magnitude: $\frac{1}{2}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{9}{16}$, $\frac{5}{9}$.

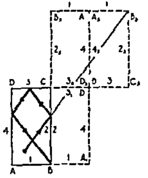
33.3. Another lattice, which may be called the Billiard Table Lattice, is useful in the geometrical solution of billiard-table problems.

*ABCD* is a billiard table, 12 ft. by 6 ft. [The full-size table is really 11 ft. $10\frac{1}{2}$ in. by 6 ft.] The cushions *AB*, *BC*, *CD*, *DA* are numbered 1, 2, 3, 4. A ball is at *P*, midway between 2 and 4 and 2 ft. from 1. Where must this ball be aimed to go into the pocket *M* after hitting 2 ? In this and all the other problems it is assumed that

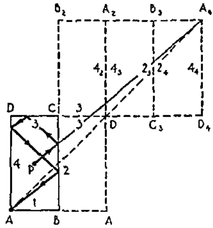
the path of the ball makes equal angles with the cushion before and after impact Reflect the billiard table in the side 2 (Fig i) The reflections of $A, D, M$ are marked $A_1, D_1, M_1$ Join $PM_1$ The point in which this line meets 2 is the point at which the ball must be aimed Why?




Ex. 1 Z 33 3 Fig i



Ex 1 Z 33 3 Fig ii



Ex. 1 Z 33.3 Fig iii

Now suppose it is required to find the path by which a ball, in a *position 2 ft from cushions 1 and 4 may go into pocket B after* hitting 2, 3, 4 Three reflections will now be necessary because three cushions are hit The reflections are indicated by the suffixes 1, 2, 3 in Fig ii Find out why these particular reflections are chosen

Join $PB_3$ This line meets 2 in the point to be aimed at Also the points in which it meets $3_1$ and $4_2$ give the points in which the actual path meets 3 and 4

A ball is at the centre of the table How must it be aimed to hit 2, 3, 4, 2 and fall into $A$?

The necessary reflections are shown in Fig. iii. The line joining $PA_4$ gives the point to be aimed at and the points where the ball hits the other cushions. The dotted line joining $D_1A_4$ divides $ABCD$ into two regions. The shot is impossible (assuming the collisions with the cushions are in the order named) if the ball is anywhere in the smaller of these regions before it is struck.

33.31. A ball is 3 ft. from 3 and 4 ft. from 4. How must it be aimed to hit 3, 4 and go into $B$?

33.32. If the ball is 2 ft. from 1 and 2 ft. from 2, how must it be aimed to hit 2, 3, 1 in that order and fall into $D$?

33.33. A ball is 2 ft. from 4 and 1 ft. from 1. How must it be aimed *to hit 2, 3, 4, 1 and pass through its starting-point again.* Find the region of the table from which this shot would be impossible, the cushions being hit in the order named.

# THE GRADIENT FUNCTION

## 2.1. The average gradient of a curve

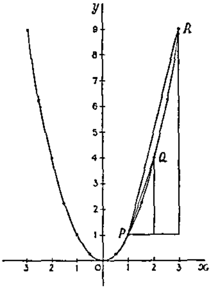



FIG. 2.1

IN Fig. 2.1, $P$, $Q$, $R$ are the points $(1, 1)$, $(2, 4)$, $(3, 9)$ on the curve $y = x^2$.

The gradient of the straight line $PQ = \dfrac{4-1}{2-1} = 3$ and the gradient of the straight line $PR = \dfrac{9-1}{3-1} = \frac{8}{2} = 4$.

Thus on a curve, in contrast to a straight line, the gradient of the line joining two points depends on the positions of the points. The gradient 4 is called the *average gradient* of the curve *from* $x = 1$ *to* $x = 3$. Similarly 3 is the average gradient from $x = 1$ to $x = 2$.

### EXERCISE 2 A

Sketch the graph of $y = x^2$, on $\frac{1}{10}$-in. paper, from $x = 0$ to $x = 3$ from the following table of values, taking the units on the $x$- and $y$-axes to be 1 in. and $\frac{1}{2}$ in. respectively.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| $y$ | 0 | 0.25 | 1 | 2.25 | 4 | 6.25 | 9 |

Calculate the average gradient of $y = x^2$

(a) from $x = 2$ to $x = 3$, (b) from $x = 2$ to $x = 2.1$,

(c) from $x = 1$ to $x = 2$, (d) from $x = 1.9$ to $x = 2$.

Draw on the graph the four lines whose gradients have just been found. (Produce the lines corresponding to (b) and (d) beyond $x = 2.1$ and $x = 1.9$ respectively so that their directions can be clearly seen.)

Draw, by eye, the tangent to the curve at $x = 2$.

The four $x$-intervals in this exercise were chosen so that each begins or ends at $x = 2$. Of the four chords corresponding to these intervals, the figure shows that two are steeper and two not so steep as the tangent at $x = 2$. Now the average gradients for the intervals (*a*) and (*c*) were found to be 5 and 3. Therefore the gradient of the tangent is between 5 and 3. Again the average gradients for the intervals (*b*) and (*d*) are 4·1 and 3·9. Therefore the gradient of the tangent is between 4·1 and 3·9, and we can say that it is 4·0 with an error which cannot be greater than

$$\frac{4\cdot1-3\cdot9}{2} = 0\cdot1.$$

We next consider how to improve upon the accuracy of this calculation. The greatest hope of success seems to lie in choosing two smaller $x$-intervals on either side of $x = 2$, e.g. from 2 to 2·01 and from 1·99 to 2. However, the calculation of the average gradients for these intervals involves tedious arithmetic and it is better to proceed algebraically as follows.

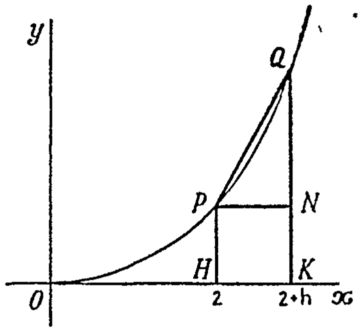


FIG. 2.2

We first calculate the average gradient of $y = x^2$ from $x = 2$ to $x = 2+h$.

$PH$ = the value of $y$ when $x = 2$, i.e. $PH = 4$.

$QK$ = the value of $y$ when $x = 2+h$,

$$\text{i.e. } QK = (2+h)^2 = 4+4h+h^2.$$

Therefore $\quad QN = QK-PH = 4+4h+h^2-4 = 4h+h^2.$

Also $\quad PN = h.$

Therefore the average gradient from $x = 2$ to $x = 2+h$ is

$$\frac{QN}{PN} = \frac{4h+h^2}{h} = 4+h.$$

By substituting different values of $h$ we may now write down the results of Ex. 2.A immediately

(*a*) if $h = 1$, the average gradient from $x = 2$ to $x = 3$ is 5.
(*b*) if $h = 0\cdot1$ „ „ $x = 2$ to $x = 2\cdot1$ is 4·1.
(*c*) if $h = -1$ „ „ $x = 2$ to $x = 1$ is 3.
(*d*) if $h = -0\cdot1$ „ „ $x = 2$ to $x = 1\cdot9$ is 3·9.

EXERCISE 2 B

1 Changing the sign of $h$, we have the average gradient from $x = 2$ to $x = 2-h = 4-h$

Prove this formula, independently, by starting with the interval $x = 2-h$ to $x = 2$

2 Write down the average gradient of $y = x^2$

(a) from $x = 2$ to $x = 2\ 01$,

(b) from $x = 1\ 99$ to $x = 2$

Between what two numbers is the gradient of the tangent at $x = 2$ now known to lie?

3 Write down the average gradient of $y = x^2$

(a) from $x = 2$ to $x = 2\ 001$

(b) from $x = 1\ 999$ to $x = 2$

What is the gradient of the tangent at $x = 2$ to 3 sig fig?

## 2.2 The gradient of the tangent

Fig 2 3 shows the curve $y = x^2$ with the tangent $t$ at $x = 2$ $l_1$ $l_2$ are the chords joining the points on the curve whose $x$ coordinates are 2, $2+h$ and $2-h$ 2 respectively We have found that the gradients of $l_1$ and $l_2$ are $4+h$ and $4-h$ and the gradient of $t$ is clearly between these two numbers By taking smaller and smaller values of $h$ we can find pairs of numbers closer and closer together between which the gradient of $t$ must lie Thus the gradient of the tangent is between

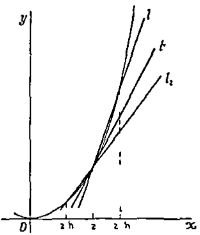



FIG 2 3

| | |
|---|---|
| 3 9 | and 4 1 |
| 3 999 | and 4 001 |
| 3 99999 | and 4 00001 |

This is the arithmetical process of section 2 1 by which we can find as close an approximation as we like to the gradient of $t$ It is clear that this approximation is very near 4

We can now go further and show that the gradient of $t$ is *exactly* 4 The difference between $4+h$ and $4-h$ is $2h$ and this can be made as small as we please by choosing $h$ small enough For example, if we want to make $2h$ smaller than a millionth we

have only to choose $h$ smaller than half a millionth. Now the numbers $4-h$ and $4+h$ always have 4 between them for all values of $h$ other than 0. Because their difference can be made as small as we like, there is no other number which is between them for all values of $h$, however small.† Hence the gradient of the tangent which is known to be between $4-h$ and $4+h$, however small $h$ may be, can only be 4 exactly.

### Exercise 2.C

1. On the curve $y = x^2$ find
   (a) the average gradient from $x = 4$ to $x = 4+h$;
   (b) the average gradient from $x = 4-h$ to $x = 4$;
   (c) the gradient of the tangent at $x = 4$.
2. On the curve $y = 3x^2$ find the average gradient from $x = 1$ to $x = 1+h$ and the gradient of the tangent at $x = 1$.

**2.3.** In practice, the argument of section 2.2 is shortened in the following manner.

In Fig. 2·4 the gradient of the chord $PQ$ joining the points whose $x$-coordinates are 2 and $2+h$, is $4+h$ (proved as before). Suppose $h$ takes smaller and smaller values getting as near to 0 as you please. Then $Q$ approaches $P$ along the curve and $PQ$ approaches the tangent at $P$. But the gradient of $PQ$, $4+h$, approaches 4. Hence we conclude that the gradient of the tangent is 4.

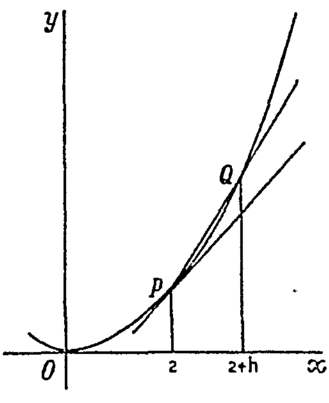


Fig. 2.4

When $h$ takes smaller and smaller values getting as near to 0 as we please, it is said to *tend to* 0 and we write $h \to 0$. This is to preserve the distinction between tending to 0 and being equal to 0. In the present example, we want $h$ to tend to 0 but we must not let $h$ equal 0. For if $h$ does equal 0, $P$ and $Q$ coincide, there is no increase in $x$ or $y$ and we cannot find any expression for the gradient.

† 3·9, for example, is not between $4-h$ and $4+h$ for all values of $h$. Take $h = 0{\cdot}01$. Then $4-h = 3{\cdot}99$ and $4+h = 4{\cdot}01$. 3·9 is not between these numbers. Now try a number greater than 4, e.g. 4·001. Take $h = 0{\cdot}0001$. Then $4-h = 3{\cdot}9999$ and $4+h = 4{\cdot}0001$. 4·001 is not between those numbers.

As $h \to 0$, $4+h \to 4$ This fixed number, to which $4+h$ approaches, is called the *limit* of $4+h$ as $h \to 0$

We are now able to calculate the gradients of the tangents which we were unable to determine satisfactorily by drawing, in section 1 19 This is done in the following example

It was remarked in section 1 19 that the fixed direction of the tangent to a curve is the instantaneous direction of the curve at its point of contact For this reason the gradient of the tangent is also called the *gradient of the curve* at the point of contact Thus the gradient of $y = x^2$ at (2, 4) is 4

EXAMPLE Find the gradient of the curve $10y = x^2$ (*a*) where $x = 5$, (*b*) where $x = 2$

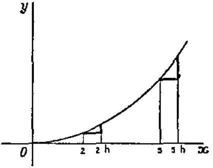



FIG 2 5

*Solution* (*a*) Since $y = x^2/10$, when $x = 5$, $y = 2\ 5$, and when $x = 5+h$,

$$y = \frac{(5+h)^2}{10}$$

$$= \frac{25+10h+h^2}{10}$$

$$= 2\ 5+h+\frac{h^2}{10}$$

The change in $y$ is

$$h+\frac{h^2}{10},$$

and the change in $x$ is $h$

Hence the average gradient from $x = 5$ to $x = 5+h$ is $1+h/10$

As $\qquad h \to 0, \qquad 1+h/10 \to 1$

Hence the gradient of the curve at $x = 5$ is 1

(*b*) Similarly, the average gradient from $x = 2$ to $x = 2+h$ is

$$\frac{(2+h)^2/10-2^2/10}{h} = \frac{4+4h+h^2-4}{10h} = \frac{4}{10}+\frac{h}{10}.$$

The limit of $4/10+h/10$ as $h \to 0$ is $4/10$

Hence the gradient of the curve when $x = 2$ is 0 4

## EXERCISE 2.D

1 Find the value of $4+10h$ when

(*a*) $h = 0\ 1$, $\qquad$ (*b*) $h = 0\ 001$,

(*c*) $h = -0\ 01$, $\qquad$ (*d*) $h = -0\ 0001$

What is the limit of $4+10h$ when $h \to 0$?

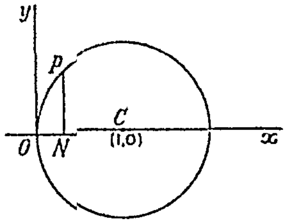
2 Find the limit of $2-5h$ when $h \to 0$

3 Find the limit of $3+8h-h^2$ when $h \to 0$

4. Find the limit of $\dfrac{(5+h)^2-25}{h}$ when $h \to 0$.

5. Find the limit of $\dfrac{2h+3h^2+4h^3}{h}$ when $h \to 0$.

6. The figure shows a circle with radius 1 and centre $(1, 0)$. Show that the equation of the circle is $x^2+y^2-2x = 0$. $P \equiv (x, y)$ is a point on the circle.




Show that $\dfrac{PN^2}{ON} = \dfrac{2x-x^2}{x}$.

As $x \to 0$, $P \to O$ and $PN \to 0$, $ON \to 0$. Find the limit of $PN^2/ON$.

7. Find the gradient of $y = 2x^2$ at $(2, 8)$.

8. Find the gradients at $(1, 1)$ and $(4, 16)$ on $y = x^2$. Find the average gradient from $(1, 1)$ to $(4, 16)$.

9. Show that the gradient of $y = 4x^2$ at $(3, 36)$ is three times the gradient at $(1, 4)$.

10. Prove that the tangent at $(3, 12)$ on $y = x^2+x$ is parallel to the chord joining the points on the curve where $x = 1$ and $x = 5$.

11. Show that the tangents at $x = 0$ and $x = 1$ on the curve $y = x^2-x$ are perpendicular.

12. Prove that the tangents at $x = 1$ and $x = -3$ on the curve $y = x^2+2x$ are equally inclined to the $x$-axis.

13. On $y = x^2$ find

(a) the gradient at $(2, 4)$;

(b) the average gradient from $(1, 1)$ to $(3, 9)$.

Illustrate by a sketch.

**2.4. Example.** Find the equation of the tangent at $(1, 3)$ to $y = 3x^2$.

*Solution.* Draw a sketch of the curve $y = 3x^2$.

Average gradient from $x = 1$ to $x = 1+h$ is
$$\frac{3(1+h)^2-3}{h} = \frac{6h+3h^2}{h} = 6+3h.$$

Let $h \to 0$. The gradient at $x = 1$ is 6.

The tangent is therefore the line through $(1, 3)$ of gradient 6.

Its equation is $\quad y-6x = 3-6 = -3,$

or $\quad y-6x+3 = 0.$

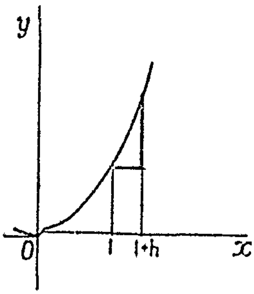



Fig. 2.6

This line cuts the axes at $(\frac{1}{2}, 0)$ and $(0, -3)$. You can verify the correctness of the work by plotting the curve $y = 3x^2$ from $x = -1$

to $x = 3$ and joining the points $(\frac{1}{2}, 0)$, $(0, -3)$ with a ruler The line should touch the curve at $(1, 3)$

EXERCISE 2 E

1 Find the equation of the tangent at $(1, 2)$ to $y = 1+x^2$, and show that it passes through the origin

2 Find the equations of the tangents at $P \equiv (-1, 1)$ and $Q \equiv (3, 9)$ on $y = x^2$ Show that the point of intersection of these tangents has the same $x$ coordinate as the middle point of the chord $PQ$ Illustrate with a figure

3 Find the equations of the tangents at $(4, 4)$ and $(-1, \frac{1}{4})$ to $4y = x^2$ Show that they are perpendicular and that they meet on $y+1 = 0$

4 Find the equations of the tangents to the curve $y = 2x-x^2$ at $(0, 0)$, $(2, 0)$ and show that they meet on $x = 1$ Draw a sketch graph and show what the result means geometrically

## 2 5. Use of gradient in a formula

When we consider an equation $y = x^2$ and its graph we may place the emphasis upon geometry or upon algebra So far, in this chapter, we have considered the graph as a geometrical curve Then $y = x^2$ is the equation of the curve and we use it, for example, to find the direction of the tangent at a point of the curve In this section we take up the algebraic point of view $y = x^2$ is a formula and the curve is its graph drawn, if at all, to give a picture of how $y$ changes when the value of $x$ changes

The formula $y = x^2$ is arranged in a suitable form for the calculation of $y$ when the value of $x$ is given In elementary algebra we call $y$ the subject of the formula Here we call $x$ the *independent variable* and $y$ the *dependent variable* The variable $x$ is described as independent because in using the formula we can give $x$ any value we choose The variable $y$ is described as dependent because in any application of the formula, the value of $y$ is calculated from, and therefore *depends* upon, the chosen value of $x$ If, in $y = x^2$, we give the independent variable, $x$, the value 3, the dependent variable, $y$ has the value 9 If we now change $x$ to 4, $y$ changes to 16, an increase of 7 We say that the change of $x$ is $+1$ and the change of $y$ is $+7$

Now let $x$ change from 3 to 2 $y$ changes from 9 to 4 We say that the change of $x$ is $-1$ and the change of $y$ is $-5$

More generally, if, when $x$ changes from $x_1$ to $x_2$ $y$ changes from $y_1$ to $y_2$, we define the *change of* $x$ as $x_2-x_1$ and the *change of* $y$ as $y_2-y_1$ (i e 'new' value minus 'old' value, in each case)

Consider, for example, the linear formula $y = 3x+7$.

Then $\qquad y_1 = 3x_1+7$

and $\qquad y_2 = 3x_2+7$,

so that $\qquad y_2-y_1 = 3(x_2-x_1)$,

or $\qquad \dfrac{y_2-y_1}{x_2-x_1} = 3.$

Since $x_1$, $x_2$ can have any values we choose, we may write this result in words

$$\frac{\textit{resulting}\text{ change of } y}{\textit{any}\text{ change of } x} = 3.$$

This constant, 3, is called the *rate of change of $y$ with $x$.* We can use it to find the change of $y$ resulting from any change of $x$.

e.g. If $x$ increases by 1, $y$ increases by 3,
if $x$ increases by 5, $y$ increases by 15,
if $x$ decreases by 2, $y$ decreases by 6.

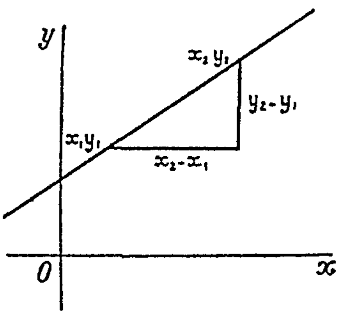



FIG. 2.7

If the graph of the formula is drawn it is a straight line (section 1.16). $(x_1, y_1)$ and $(x_2, y_2)$ become two points on the graph and $(y_2-y_1)/(x_2-x_1)$ is the gradient of the straight line. (Fig. 2.7.)

*The rate of change of $y$ with $x$ is the gradient of the straight line graph of the formula.*

For the linear formula $y = 5-2x$,

$$\frac{y_2-y_1}{x_2-x_1} = -2.$$

The rate of change of $y$ with $x$ is negative.

If the change of $x$ is $+1$, the change of $y$ is $-2$;
if the change of $x$ is $-1$, the change of $y$ is $+2$.

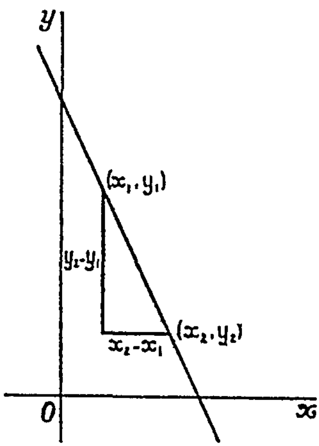



FIG. 2.8

The graph is a straight line with negative gradient, i.e. a falling line. (Fig. 2.8.)

## EXERCISE 2 F

1 If $y = 2x - 1$ and $z = 12 - \frac{1}{2}x$, find the changes in $y$ and $z$ when $x$ changes from 2 to 6 What are the rates of change of $y$ and $z$?

2 If $y = 1.5x - 11$ and $z = 2 - 7x$, find the changes of $y$ and $z$ when $x$ changes from 2 to 5 Find the rates of change of $y$ and $z$ with $x$

3 $y = 14 - 5x$ If $x$ changes from $a$ to $a+h$, find the change of $y$ Find the rate of change of $y$ with $x$

4 If $2y = 3 - 7x$, what is the rate of change of $y$ with $x$?

5 $y = 16 + 8x$ What is the rate of change of $y$ with $x$? For what value of $x$ is $y$ zero? For what values of $x$ is $y$ positive?

6 $y = 9 - \frac{3}{2}x$ For what values of $x$ is $y$ positive?

7 Find the value of $y$ when $x = 3$ in the formula $y = (1+4x)/2$ For what values of $x$ is $y$ greater than this value?

8 In a certain formula when $x = 4$, $y = 3$, and when $x = 7$, $y = 15$ If the formula is linear, what is the rate of change of $y$ with $x$? Without finding the formula find $y$ (*a*) when $x = 6$, (*b*) when $x = 10$

9 In a certain formula when $x = 2$, $y = 19$ and when $x = 12$, $y = 4$ If the formula is linear what is the rate of change of $y$ with $x$? Find $y$ (*a*) when $x = 4\frac{1}{2}$, (*b*) when $x = 0$

10 The weight, $W$ lb, of a round steel bar 1 in in diameter and $x$ ft long is given by the formula $W = 2.67x$ What is the rate of increase of weight with length?

A rod is 10 ft long and it is proposed to replace it by one 8 in longer What will be the increase in weight?

11 The length, $l$ in, of a steel rail at temperature $t°$ F, is approximately given by $l = 720 + 0.005\,t$ Find the rate of increase of length with temperature In Britain the annual range of shade temperature may amount to 85° F. What change in the length of the rail must be allowed for in this country if it is shaded from the direct rays of the sun?

12 The petrol, $G$ gallons, in an aircraft $t$ hours after take off is given by $G = 800 - 75t$ What is the consumption of petrol per hour? With what quantity of petrol does the aircraft take off?

13 The horse power, $H$, required to lift $G$ gallons of water per min to a certain height is given for different values of $G$ in this table

| $H$ | 0.15 | 1.50 | 3.03 | 6.06 |
|---|---|---|---|---|
| $G$ | 50 | 500 | 1000 | 2000 |

*Show that the law connecting $G$ and $H$ is linear What is the rate* at which $H$ increases with $G$, to 1 sig fig?

## 2.6. Constant velocity

One of the commonest examples of a rate is the velocity of a moving body.

Suppose a lorry is travelling along a straight road and at a certain instant passes an A.A. box. Suppose also that $t$ min. later its distance from the box is $s$ miles where $s = \frac{1}{2}t$. This formula is called the *distance-time* formula for the motion of the lorry. It is assumed that $s$ is measured positively in a definite sense along the road, and, negatively, in the opposite sense.

The formula is linear. The rate of change of $s$ with $t$ is $\frac{1}{2}$. The distance of the lorry from the box increases at the constant rate

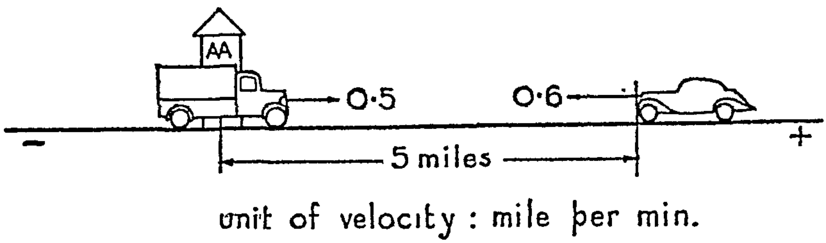



FIG. 2.9

of $\frac{1}{2}$ mile in every minute. We say that the *velocity* of the lorry is $\frac{1}{2}$ mile per min.

Now consider a car moving on the road so that $t$ min. after the lorry was at the A.A. box the car is $s$ miles from the box, where $s$ and $t$ are connected by the formula $s = 5 - 0{\cdot}6t$, and $s$ is measured positively in the same sense as before. The rate of change of $s$ with $t$ is $-0{\cdot}6$ mile per min. Hence $s$ decreases with $t$ and the car is moving in the opposite direction to the lorry.

When $t = 0$, the lorry is at the A.A. box and the car is $+5$ miles from it. The relative positions and directions of travel at this instant are shown in Fig. 2.9.

The velocities of the lorry and car are $0{\cdot}5$ and $-0{\cdot}6$ mile per min. respectively. We might want to decide which vehicle is moving faster. This we do by comparing their *speeds* which are $0{\cdot}5$ and $0{\cdot}6$ mile per min. The car is moving faster.

The *velocity* of a body has magnitude and direction; the magnitude of the velocity is called the *speed*. The direction of motion in which the velocity is reckoned positively is the direction in which the variable representing distance increases.

When $t = 10$, the distance-time formula for the motion of the car gives $s = -1$. This shows that the car has passed the A.A. box and is 1 mile beyond it.

## Exercise 2 G

1 On a straight road running east and west there is a milestone The distance time formulae for 4 vehicles travelling on this road are given below $s$ represents distance from the milestone in yards and is reckoned positively towards the east $t$ is time from a certain instant in minutes

Vehicle $A$ $s = 50+2500t$, Vehicle $B$ $s = 3000t-100$,
Vehicle $C$ $s = 200-4000t$, Vehicle $D$ $s = 20-2000t$

(i) *In what direction is $A$ travelling?*
(ii) Give the velocity of each vehicle in yd per min Which is travelling the fastest?
(iii) Name the vehicles which are travelling from east to west
(iv) Give the distance of each vehicle east or west of the milestone at $t = 0$
(v) Give the order in which the vehicles pass the milestone
(vi) Give the distance east or west of the milestone at which $B$ and $D$ pass each other

2 A train leaves Paddington and after passing Reading its distance time formula is $s = 60t-10$, where $s$ is distance in miles from Paddington and $t$ is time in hours from Paddington What is the speed of the train from Reading? How long does it take from Paddington to Swindon (77 miles from Paddington) if this speed is maintained all the way from Reading?

3 As a train passes a quarter mile post a stop watch is started and the next 3 quarter mile posts are passed after 18 sec, 36 sec, 54 sec *Show that the speed is constant and find it in m p h*

4 A train leaves a station and passes a signal, then a signal box 220 yd farther on, and it enters a tunnel after travelling a further 550 yd The times of passing are 2 min 05 sec, 2 min 15 sec, and 2 min 40 sec from leaving the station Show that the speed of the train is constant from the signal to the tunnel and find it

5–7 In the following distance time formulae $s$ is in ft and $t$ in sec Find the velocity in each case

5 $s = 17-8t$ 6 $2s = 5+17t$ 7 $s = \frac{1}{4}(18t-5)$

8 A meteorologist releases a balloon to determine the temperature and pressure of the upper air The height of the balloon at various times is

| Height above sea level (ft) | 211 | 713 | 2721 | 5231 |
|---|---|---|---|---|
| Time (min) | 0 | 1 | 5 | 10 |

Show that the balloon rises at a constant rate and find this rate

## 2.7. Variable Rate

Now consider the equation $y = x^2$ as a formula. When $x$ changes from 1 to 3, $y$ changes from 1 to 9:

$$\frac{\text{change of } y}{\text{change of } x} = \tfrac{8}{2} = 4.$$

When $x$ changes from 1 to 2, $y$ changes from 1 to 4:

$$\frac{\text{change of } y}{\text{change of } x} = \tfrac{3}{1} = 3.$$

Hence the rate of change of $y$ with $x$ is not constant.

If $y$ changes from $y_1$ to $y_2$ when $x$ changes from $x_1$ to $x_2$, the value of the fraction $(y_2-y_1)/(x_2-x_1)$ is called the *average rate of change* of $y$ with $x$ from $x_1$ to $x_2$.

Therefore for the formula $y = x^2$, the average rate of change of $y$ with $x$ from $x = 1$ to $x = 3$ is 4. Similarly, the average rate of change of $y$ from $x = 1$ to $x = 2$ is 3.

Now consider the change in $y$ when $x$ changes from 1 to $1+h$.

When $x_1 = 1$, $y_1 = 1$, and when $x_2 = 1+h$, $y_2 = 1+2h+h^2$

$$\frac{y_2-y_1}{x_2-x_1} = \frac{2h+h^2}{h}$$
$$= 2+h.$$

When $h \to 0$, $2+h \to 2$.

2 is defined to be the *rate of change* of $y$ with $x$ *at* $x = 1$.

If the graph of the formula is drawn, the average rate of change from $x = 1$ to $x = 3$ is the gradient of the chord joining the points (1, 1) and (3, 9). Similarly the average rate of change from $x = 1$ to $x = 1+h$ is the gradient of the chord joining the points (1, 1) and $(1+h, 1+2h+h^2)$.

Since we find the rate of change at $x = 1$ by letting $h \to 0$, this rate is the gradient of the tangent to the graph at $x = 1$. In general, if $x$ and $y$ are connected by a formula, the rate of change of $y$ with $x$ at $x = x_1$ is the gradient of the graph of the formula at $x = x_1$.

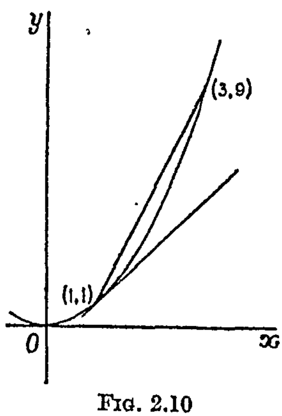



Fig. 2.10

## Exercise 2 H

1 If $y = 10x^2$ find

(i) the average rate of change of $y$ with $x$ from $x = 5$ to $x = 5+h$,

(ii) the average rate of change of $y$ with $x$ from $x = 5$ to (a) $x = 5.1$, (b) $x = 5.01$, (c) $x = 5.001$,

(iii) the rate of change of $y$ with $x$ at $x = 5$

2–4 Find the average rate of change of $y$ with $x$ for the stated change in $x$ and the rate of change of $y$ with $x$ for the stated value of $x$ in

2 $y = 3x^2$, $x = 2$ to $x = 2+h$, $x = 2$

3 $y = 1+\frac{1}{2}x^2$, $x = 4$ to $x = 4+h$, $x = 4$

4 $y = 4x-x^2$, $x = 5$ to $x = 5+h$, $x = 5$

5 The force, $P$ lb wt, on a plane surface of area 1 sq ft held perpendicular to a current of water flowing at $v$ f p s, is given by the formula $P = 1.8v^2$ What is the average rate of increase of force with the velocity of the current from $v = 2$ to $v = 5$? What is the rate of increase of $P$ with $v$ when $v = 3$?

6 You are given two formulae, $y = x^2/2$ and $z = 4x-8$

(i) Find the rate of increase of $y$ with $x$ when $x = 4$

(ii) Find the constant rate of increase of $z$ with $x$

(iii) Show that $y = z$ when $x = 4$

(iv) Sketch the graphs of the formulae on the same axes

(v) What is the relationship of the straight line to the curve?

7 If $y = x^2/10$, find the rate of increase of $y$ with $x$ when $x = 5$ If $y$ increases according to this formula as $x$ increases from $x = 0$ to $x = 5$ and after that the rate of increase of $y$ with $x$ remains constant at its value when $x = 5$, find the formula giving $y$ in terms of $x$ when $x$ is greater than 5

8 Find the rate of increase of $y$ with $x$ when $x = 2$, if the formula connecting them is $y = 6-x^2$ As $x$ increases through the value 2, is $y$ increasing or decreasing?

9 The lengths of cod and grey mullet are given at various ages in the following table †

| Age (yrs) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Length of cod (cm) | 18 | 36 | 55 | 68 | 79 | 89 |
| Length of grey mullet (cm) | 21 | 36 | 46 | 51 | 53 | 55 |

Draw graphs on the same axes What is the rate of growth of each fish (a) at 3 yrs, (b) at 5 yrs? (c) At approximately what age does the rate of growth of grey mullet become constant? (d) What is the average rate of growth of each fish during the 5 yrs?

† D'Arcy Thompson, *Growth and Form*, p 209

10. The mean height of the barometer in in. of mercury at various heights in ft. above sea-level is shown in the following table:

| | | | | | | |
|---|---|---|---|---|---|---|
| Barometer height in in. of mercury . | 30 | 28 | 26 | 24 | 22 | 20 |
| Height above sea-level (ft.) . . | 0 | 1600 | 3500 | 5500 | 7800 | 10300 |

Draw the graph and determine from it the average rate of fall of the barometer in in. of mercury per 1000 ft.

(*a*) from 0 to 2,000 ft., (*b*) from 7,000 to 10,000 ft.

Find as accurately as you can the rate of fall at 5,000 ft.

## 2.8. Variable Velocity

If the distance-time formula of a body moving in a straight line is not linear (e.g. $s = 5+\frac{1}{2}t^2$), the velocity of the body is not constant. The average rate of increase of $s$ with $t$ in any interval of time is called the *average velocity* in that interval of time. The rate of change of $s$ with $t$ at any instant is called the *velocity* of the body at that instant.

If the graph of the distance-time formula is drawn, the velocity at any time is the gradient of the graph at the corresponding point.

Example. If the distance-time formula is $s = 5+\frac{1}{2}t^2$, find the velocity when $t = 4$.

*Solution.* The average velocity from $t = 4$ to $t = 4+h$ is

$$\frac{\text{change of } s}{\text{change of } t} = \frac{5+\frac{1}{2}(4+h)^2-(5+8)}{h}$$

$$= \frac{5+8+4h+\frac{1}{2}h^2-(5+8)}{h}$$

$$= 4+\tfrac{1}{2}h.$$

When $h \to 0$, $\qquad 4+\frac{1}{2}h \to 4.$

The velocity at $t = 4$ is 4.

*Note.* The units of $s$ and $t$ are not stated in this example and the answer is left as a number. We might give the answer in the form 4 units of distance per unit of time, but this is not customary. In any question where the units are given, the unit in which the answer is measured must be stated. For example, in this question, if $s$ were given in ft. and $t$ in sec. the answer would be 4 f.p.s.

## EXERCISE 2 J

*In this exercise the bodies are supposed to move in a straight line*

1 If the distance time formula is $s = 2+5t^2$, find the velocity when $t = 1$ Find the average velocity from $t = 0$ to $t = 4$

2 If the distance, $s$ yd, travelled by a train $t$ min after starting is given by $s = 50t^2$, find the velocity of the train in yd per min after 10 min

3 If the distance time formula is $s = t^2-2t$, show that the body is travelling 3 times as fast when $t = 4$ as when $t = 0$ and in the opposite direction

4 By means of electrical apparatus the times at which a runner passes various distances from his starting point can be measured to $\frac{1}{100}$ sec The following table shows the corresponding times and distances for the first 20 yd of an actual run †

| Distance (yd) | 0 | 1 | 3 | 6 | 10 | 15 | 20 |
|---|---|---|---|---|---|---|---|
| Time (sec) | 0 | 0 66 | 1 04 | 1 50 | 2 00 | 2 56 | 3 08 |

Draw the distance time graph (suitable scales are 1 in = 5 yd and 1 in = 1 sec) and find approximately the velocity (*a*) 1 sec after the start, (*b*) 15 yd from the start (*c*) Does it appear from the times at 10, 15, 20 yd that the runner attains his top speed during the first 20 yd ? (*d*) If the total length of the race was 100 yd and the runner covered the remainder of the distance at his average speed from 15 to 20 yd, in what time would he run the 100 yd ?

5 The distance of a body from its starting point after various times is given by the following table

| $s$ (ft) | 0 | 8 | 12 | 12 | 8 | 0 | −12 |
|---|---|---|---|---|---|---|---|
| $t$ (sec) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

Draw a graph and find (i) the average velocity during first 3 sec, (ii) the velocity at $t = 2$, (iii) the time at which the body is instantaneously at rest, (iv) the velocity with which the body passes through its starting point again

6 The following table gives the distances of a car from a road junction at various times Draw a graph and find its speed (*a*) at 30 sec, (*b*) at 100 sec, as accurately as you can (*c*) Find the average speed from 60 to 90 sec

| Distance (yd) | 688 | 352 | 149 | 44 | 1 7 | 0 |
|---|---|---|---|---|---|---|
| Time (sec) | 0 | 30 | 60 | 90 | 130 | 150 |

† A V Hill, *Living Machinery* p 248

## 2.9. The gradient function

We have now seen how the gradient may be used to draw a tangent to a curve or to find a rate of change in a formula. We are somewhat handicapped in these applications at present by the tedious work of finding the gradient at each point or value where it is required. Our next object must be to show how this labour can be very greatly reduced.

Examination of the worked examples and exercises of this chapter shows that we have already worked out the gradients of $y = x^2$ for the following values of $x$.

| $x$ | $-1$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Gradient of $x^2$ | $-2$ | 2 | 4 | 6 | 8 |

This table suggests that the gradient of $x^2$ for any value of $x$ is twice that value of $x$. We now show that this is true for all values of $x$.

Let $x_1$ be any value of $x$ and as $x$ changes to $x_1+h$, let $y$ change from $y_1$ to $y_2$.

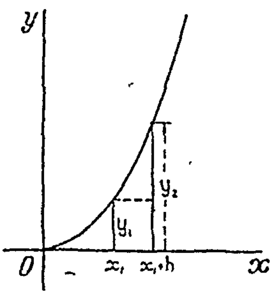


FIG. 2.11

Then $\qquad y_1 = x_1^2$

and $\qquad y_2 = (x_1+h)^2$

$\qquad\qquad = x_1^2+2x_1h+h^2.$

Therefore $\quad y_2-y_1 = 2x_1h+h^2.$

The average gradient from $x = x_1$ to $x = x_1+h$ is

$$\frac{2x_1h+h^2}{h} = 2x_1+h \to 2x_1 \quad \text{when} \quad h \to 0.$$

Therefore the gradient of $y = x^2$ when $x = x_1$ is $2x_1$.

We are accustomed to denote the coordinates of *any* point on $y = x^2$ as $(x, y)$. The gradient at this point is $2x$.

For the present we shall call $2x$ the *gradient function* of $x^2$. It enables us to *write down*, by substitution, the gradient at any given point of the curve.

e.g. the gradient at $(5, 25)$ is $2\times 5 = 10$,

the gradient at $(-2, 4)$ is $2\times -2 = -4$.

The word *function* is used in mathematics to denote a number which depends upon another variable number. Thus $x^2$, $x^3-2x+1$, $1/x$, $\sin x$, $\log x$, are all called functions of $x$.

If $y$ and $x$ are connected by a formula, the rate of change of $y$ with $x$ for a given value of $x$ is the gradient of the graph of the formula at the corresponding point. Hence the rate of change of $y$ with $x$ can be written down immediately from the gradient

function For example, if $y = x^2$, the rate of change of $y$ with $x$ when $x = 7$ is $2 \times 7 = 14$

### Exercise 2.K

1 Find the gradient function of $x^2/5$ Find the equation of the tangent to $5y = x^2$ at $(5, 5)$

2 Verify that the chord joining the points $(-1, 1)$ and $(5, 25)$ of the curve $y = x^2$ is parallel to the tangent at $(2, 4)$

3 Find the equations of the tangents to $y = x^2$ at $(2, 4)$ and $(4, 16)$ Verify that the $x$ coordinate of their point of intersection equals the $x$ coordinate of the middle point of the chord joining their points of contact

4 What is the gradient of $y = x^2$ at $x = 2\frac{1}{2}$ ? Find the coordinates of the point on the curve at which the tangent is perpendicular to the tangent at $x = 2\frac{1}{2}$ Find the equation of this tangent

5 If $y = x^2$, show that the rate of increase of $y$ with $x$ when $x = 100$ is 50 times its rate of increase when $x = 2$ For what value of $x$ is the rate of decrease of $y$ one half its rate of increase when $x = 2$ ?

6 Find the gradient function of $y = 0\ 06x^3$

The air resistance, $R$ lb wt , on a certain car travelling at $v$ m p h is given by the formula $R = 0\ 06v^3$ Find the rate of increase of $R$ with $v$ when the velocity is 20 m p h At what velocity is the rate of increase of $R$ with $v$, 6 lb wt per m p h ?

7 Find the gradient function of $y = 16x^2$

When a stone falls freely from rest the distance fallen, $s$ ft , is given in terms of the time of fall $t$ sec , by the formula $s = 16t^2$

(i) Find the velocity after it has been falling for 3 sec

(ii) Find a formula for the velocity after falling for $t$ sec

(iii) After how long will the velocity be 400 f p s ?

(iv) How long will the stone take to fall 256 ft ?

(v) What will be its velocity after falling 256 ft ?

8 Find the gradient function of $4x^2$ $P$ is the point $(2, 16)$ on $y = 4x^2$ $PN$ is drawn parallel to the $x$ axis to meet the $y$ axis at $N$ The tangent at $P$ meets the $y$ axis at $T$, and $O$ is the origin Show that $TO = ON$

## 2.10 The gradient function of other powers

We also need to write down quickly the gradient function of other powers of $x$ We consider next the function of cubes,

$$y = x^3$$

Sketch the graph of $y = x^3$ and let $P$ be any point of the curve with the general $x$-coordinate, $x$. As $x$ changes from $x$ to $x+h$, the change of $y$ is $(x+h)^3-x^3$.

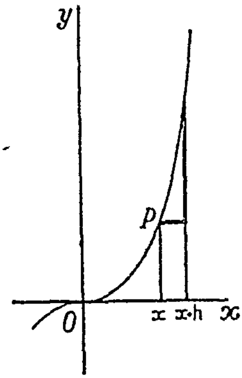


FIG. 2.12

Now
$$\begin{aligned}(x+h)^3 &= (x+h)^2(x+h)\\ &= (x^2+2xh+h^2)(x+h)\\ &= x^3+3x^2h+3xh^2+h^3.\end{aligned}$$

$$\begin{array}{l} x^2+2xh+h^2 \\ x\ +h \\ \hline x^3+2x^2h+\ xh^2 \\ \qquad\ \ x^2h+2xh^2+h^3 \\ \hline x^3+3x^2h+3xh^2+h^3. \\ \hline \end{array}$$

Therefore the change of $y$ is

$x^3+3x^2h+3xh^2+h^3-x^3 = 3x^2h+3xh^2+h^3$.

Therefore the average gradient from $x$ to $x+h$ is

$$\frac{3x^2h+3xh^2+h^3}{h} = 3x^2+3xh+h^2.$$

Let $h \to 0$. Then $3x^2+3xh+h^2 \to 3x^2$ and the gradient at $P$ is $3x^2$.

*Therefore the gradient function of* $x^3$ *is* $3x^2$.

In the same way, we can find the gradient functions of $x^4$, $x^5$,..., but we then require the results of multiplying out $(x+h)^4$, $(x+h)^5$,.... If the Binomial Theorem is known, these products may be written down and Ex. 2.L may be worked immediately, the remainder of this section being omitted.

Otherwise, we must first note that the *form* of the products can be predicted. We know that

$$\begin{aligned}(x+h)^2 &= x^2+2xh+h^2\\ (x+h)^3 &= x^3+3x^2h+3xh^2+h^3,\end{aligned}$$

and we can see that

$$\begin{aligned}(x+h)^4 &= x^4+(\ \ )x^3h+(\ \ )x^2h^2+(\ \ )xh^3+h^4\\ (x+h)^5 &= x^5+(\ \ )x^4h+(\ \ )x^3h^2+(\ \ )x^2h^3+(\ \ )xh^4+h^5,\end{aligned}$$

where certain numbers must be written in each of the brackets. A glance through the working by which we obtained the gradient function of $x^3$ shows that we only need the first of these unknown numbers. After division by $h$ to find the average gradient, all the terms after the first contain at least $h^1$ and therefore tend to 0 when $h \to 0$. We now find the first unknown number in the expansion of $(x+h)^4$ and $(x+h)^5$.

$$(x+h)^4 = (x+h)^3(x+h) = (x^3+3x^2h+3xh+h^3)(x+h)$$
$$= x^4+4x^3h+$$

$$\begin{array}{l} x^3+3x^2h+3xh+h^3 \\ \underline{x+h \qquad\qquad\qquad} \\ x^4+3x^3h+ \\ \quad\;\; \underline{x^3h+} \\ \underline{x^4+4x^3h+} \end{array}$$

$$(x+h)^5 = (x+h)^4(x+h) = (x^4+4x^3h+ \quad )(x+h)$$
$$= x^5+5x^4h+$$

$$\begin{array}{l} x^4+4x^3h+ \\ \underline{x+h \qquad\quad} \\ x^5+4x^4h+ \\ \quad\;\; \underline{x^4h+} \\ \underline{x^5+5x^4h+} \end{array}$$

### Exercise 2 L

1–2 Use $(x+h)^4 = x^4+4x^3h+$ and $(x+h)^5 = x^5+5x^4h+$ to find the gradient function of

1 $x^4$ 2 $x^5$

3–4 Find the gradient function of

3 $x$ 4 1

5 Complete the following table

| | | | | | | |
|---|---|---|---|---|---|---|
| Function | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x$ | $1(=x^0)$ |
| Gradient function | | | | | | |

**2.11.** Example Find the gradient function of $1/x^2$

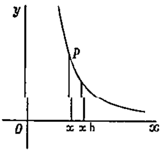



Fig 2 13

*Solution* Let $y = 1/x^2$ and sketch the graph

As $x$ changes from $x$ to $x+h$

$$\text{the change of } y = \frac{1}{(x+h)^2} - \frac{1}{x^2}$$
$$= \frac{x^2-(x+h)^2}{x^2(x+h)^2} = \frac{-2xh-h^2}{x^2(x+h)^2}$$

Average gradient from $x$ to $x+h$

$$= \frac{\text{change of } y}{\text{change of } x} = \frac{-2xh-h^2}{hx^2(x+h)^2} = \frac{-2x-h}{x^2(x+h)^2}$$

Let $h \to 0$

The gradient at $P = \dfrac{-2x}{x^4} = -\dfrac{2}{x^3}$ if $x \neq 0$

## Exercise 2.M

1. Find the gradient function of $1/x$.
2. Find the gradient function of $1/x^3$.
3. *Guess* the gradient function of $x^6$ and $x^7$.
4. Write $1/x$, $1/x^2$, $1/x^3$ as $x^{-1}$, $x^{-2}$, $x^{-3}$. Can you *write down* the gradient functions of these three functions by the method of No. 3?
5. Find the gradient function of $10x^3$. Compare with the gradient function of $x^3$.
6. Find the gradient function of $x^2+x^3$. In what way can the gradient function of $x^2+x^3$ be found from the gradient functions of $x^2$ and $x^3$?
7. Find the gradient function of $x-1/x$. In what way can the gradient function of $x-1/x$ be found from the gradient functions of $x$ and $1/x$?
8. Assuming the gradient functions of $x$ and $x^2$, *write down* the gradient function of $5+2x-3x^2$.

**2.12.** From particular cases worked out in sections 2.10, 2.11, and Ex. 2.M we make the following generalizations. These enable us to write down the gradient function of any function which can be expressed as a sum (or difference) of positive or negative integral powers of $x$.

(1) The gradient function of $x^n$ is $nx^{n-1}$ if $n$ is a positive or negative integer.

(2) The gradient function of $cx^n$, where $c$ is a constant, is $cnx^{n-1}$.

(3) The gradient function of a sum (or difference) of powers of $x$ is the sum (or difference) of the gradient functions of the powers taken separately.

(4) If the value of a function is constant, its gradient is 0 for all values of $x$. (This is clear from its graph which is a straight line parallel to the $x$-axis.)

Conversely, if a function has zero gradient for all values of $x$, it is a constant.

Example 1. Find the gradient function of $(x^2-2)^2/x^2$.

*Solution.* Write the function as a sum of powers,

$$\frac{(x^2-2)^2}{x^2}=\frac{x^4-4x^2+4}{x^2}=x^2-4+4x^{-2}.$$

[Gradient function of $x^2=2x$; Gradient function of $-4=0$; Gradient function of $4x^{-2}=-8x^{-3}=-8/x^3$.]

The required gradient function is $2x-8/x^3$.

In practice, the work shown in brackets is not written down.

EXAMPLE 2 The distance time formula of a body moving in a straight line is $s = \frac{1}{6}t^3 - 3t^2 - 14t$ and the motion starts when $t = 0$ Find the velocity after time $t$ and the time at which the body is instantaneously at rest

*Solution* Let the velocity of the body after time $t$ be $v$ Then $v$ is the rate of change of $s$ with $t$ at time $t$

Hence $v = \frac{1}{2}t^2 - 6t - 14$

To find when the body is instantaneously at rest put $v = 0$

Then
$$0 = \tfrac{1}{2}t^2 - 6t - 14,$$
$$0 = t^2 - 12t - 28,$$
$$0 = (t-14)(t+2),$$
$$t = 14 \text{ or } -2$$

It is usual to ignore negative solutions since these denote times earlier than $t = 0$ when the motion is supposed to start

The body is at rest when $t = 14$ The physical explanation of this is that the body reverses its direction of motion at $t = 14$ The formula for $v$ gives $v = -7\frac{1}{2}$ when $t = 13$ and $v = 8\frac{1}{2}$ when $t = 15$ The different signs indicate that the body is travelling in opposite directions at these two instants

*Note* If the graph of the distance time formula is drawn, the velocity at time $t$ is the gradient of the graph for the general value, $t$, of the time *The velocity at time $t$ is the gradient function of the distance, $s$*

## EXERCISE 2 N

1–41 Write down the gradient functions of these functions of $x$

1 $3x^2$ 2 $\frac{1}{12}x^3$ 3 $3$ 4 $8x^7$

5 $\frac{1}{10}x^{10}$ 6 $4/x$ 7 $1/2x^2$ 8 $-1/x^4$

9 $-1/3x^3$ 10 $2x-x^2$ 11 $3+8x$

12 $5+4x^3$ 13 $x+\frac{1}{4}x^2$ 14 $1-2x^3$

15 $1+2x-3x^3$ 16 $\frac{1}{2}x^2+\frac{1}{3}x^3$ 17 $(3x^2-4x^3)/6$

18 $\frac{1}{5}(1-x+2x^2)$ 19 $\frac{2}{3}x^3-4x+5$ 20 $-1/x^2$

21 $3/x^3$ 22 $1/4x^4$ 23 $x-1/x$

24 $x^2+1/x^2$ 25 $2x^3-3/x^2$ 26 $3x^2+2/x^3$

27 $x(3+4x)$ 28 $(x+1)^2$ 29 $(x^2+1)/x$

30 $(1+x)(1+x^2)$ 31 $(1+x)(1-x)/x$

32 $(1-2x)^2/x$ 33 $x(x-2)(x+2)$

34 $1-2x^3+x^6$ 35 $5x^2-2/x^3$ 36 $3x^3+4x^2+5x$

37 $(5-10x+x^5)/5$ 38 $x(x^9-x^{-10})$

39 $(2x-3/x)^2$ 40 $\pi(x^2-x^4)$ 41 $\pi(x^4-3)/x$

42–5 Find the velocity, at time $t$ of a body travelling in a straight line with the given distance time formula

42 $s = 10t^2-4t+3$ 43 $s = 5-4t+t^2/8$

44 $s = \frac{1}{2}(t-1)^2$ 45 $s = 4t^3-12t$

46. The safe load, $T$ tons, which may be put on a steel beam $s$ ft. long, $7\frac{1}{2}$ in. wide, and 24 in. deep, is given by the formula $T = 1125/s$.
Find the rate of change of $T$ with $s$ when (a) $s = 10$, (b) $s = 50$.

47. A body moves along the $x$-axis starting from the origin $O$, at $t = 0$. After $t$ min. its position is given by $x = t^3 - 3t^2$.

(a) Find its velocity at $t = 1$ and $t = 4$. Is it moving in the same or opposite directions at these two instants?
(b) Show that the body starts from rest.
(c) When is it again instantaneously at rest? What is its position then?
(d) When is it again at $O$? What is its velocity then?

48. The distance-time formula for the motion of a body in a straight line is $s = 27t - \frac{1}{4}t^3$ where $s$ is measured in ft. and $t$ in min. The start occurs at $t = 0$.

(i) Find the velocity at $t$ min.
(ii) Find the velocity at (a) 2 min., (b) 10 min.
(iii) Find the velocity at the start.
(iv) How long after the start is the velocity instantaneously 0?
(v) Fill in the blanks in the following statement. The body starts with velocity ______ ft. per min. and moves with decreasing velocity until it is instantaneously at rest after ______ min. when its distance from the start is ______ ft. It then reverses its direction of motion and moves with increasing speed reaching the start again when $t$ is the positive root of the equation $t^2 = 108$ (obtained by putting $t = 0$ in the distance-time formula. Verify this.) It passes the starting-point with velocity ______ ft. per min.

49. A ball is thrown vertically upwards and its height, $H$ ft., above ground after $t$ sec. is given by $H = 80t - 16t^2$.

(i) Show that it reaches the ground again after 5 sec.
(ii) Find its velocity after $t$ sec.
(iii) Find its velocity after 1 sec., 3 sec. Explain the meaning of the signs of these velocities.
(iv) Find the velocity with which it is thrown up.
(v) How long after the start is the ball instantaneously at rest?
(vi) What is the greatest height reached by the ball?

50. A stone is thrown vertically downwards from the edge of a cliff 400 ft. high. After $t$ sec. its height, $H$ ft. above the level of the foot of the cliff, is given by $H = 400 - 36t - 16t^2$. Show that it reaches the ground after 4 sec. Find its velocity after $t$ sec. Find the velocity with which it reaches the ground.

51 Find the point on $y = x^3$ at which the gradient is 6 Find the equation of the tangent at this point

52 Find the equation of the tangent to $y = \frac{1}{6}x^2$ which is parallel to $3y-2x = 0$ [First find the point of contact ]

53 Find the equation of the tangent to $y = x^2+4x+5$ which is parallel to (a) $y+2x = 0$, (b) the $x$ axis

54 Find the coordinates of the point on $y = 2x^4$ at which the tangent is perpendicular to the tangent at $(\frac{1}{2}, \frac{1}{8})$

55 Find the equation of the tangent to $3y = x+x^2$ which is perpendicular to the tangent at the point where $x = 4$

## *2.13. Curve drawing and curve sketching*

Suppose we want to draw an accurate graph of $y = x^2+4x$ We make a table of values and plot points, joining them by a smooth line

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|---|
| $4x$<br>$x^2$ | $-20$<br>25 | $-16$<br>16 | $-12$<br>9 | $-8$<br>4 | $-4$<br>1 | 0<br>0 | 4<br>1 | 8<br>4 |
| $y$ | 5 | 0 | $-3$ | $-4$ | $-3$ | 0 | 5 | 12 |

Accuracy may be improved by plotting points which lie closer together In the present example we might take values of $x$ between the integral values given in the table In this way we obtain a graph which may be used to read off corresponding values of $x$ and $y$

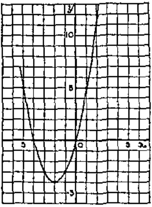


Fig 2 14

But a graph has another very important use with which we are more concerned here It gives at a glance a general picture of the way in which $y$ changes as $x$ changes For this purpose a sketch is all that is needed The rise and fall of $y$ must be shown correctly, but the graph need not be so accurate that values can be read from it In fact, it need not be drawn on squared paper at all

For example, suppose we want to draw a sketch of $y = x^2+4x$ Denote the gradient function by $g$

Then $g = 2x+4$

*Sketch the graph of $g$ against $x$*, this is particularly easy, since it is a straight line (Fig 2 15)

This graph shows that $g$ is positive when $x$ is greater than $-2$, and negative when $x$ is less than $-2$. Therefore the graph of $y$ rises when $x$ is greater than $-2$ and falls when $x$ is less than $-2$. The value of $y$ when $x = -2$ is $-4$.

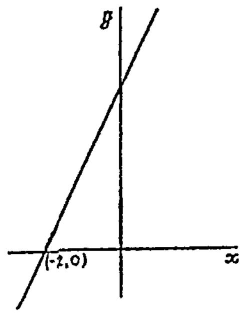



FIG. 2.15

These facts are sufficient to enable a sketch to be drawn. An additional guide is the fact that the graph passes through the origin. The graphs are most conveniently drawn together, with parallel $x$-axes and $g$- and $y$-axes in line (Fig. 2.16). Full advantage may then be taken of the information provided by the graph of the gradient function. If $g$ is positive, then the bigger it is the more steeply the $(x, y)$-curve rises and if $g$ is negative, the bigger its numerical value is the more steeply the $(x, y)$-curve falls.

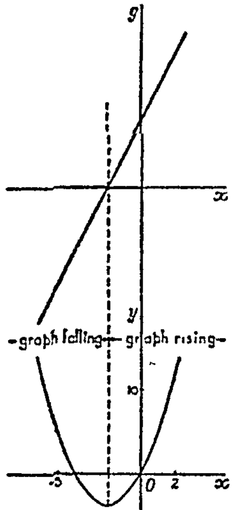



FIG. 2.16

As an example of the useful information about a function which may be deduced from quite a rough sketch graph, note that the points in which the curve crosses the $x$-axis are found by solving $x^2+4x = 0$. The solutions are $x = 0$ and $x = -4$. Then the graph shows that $x^2+4x$ is negative for values of $x$ between $-4$ and $0$ and positive for values of $x$ greater than $0$ or less than $-4$.

Further help in curve sketching is sometimes given by symmetry. For example, the graph of $y = x^2$ is such that if $(x_1, y_1)$ is a point on it, so is $(-x_1, y_1)$ [if $y_1 = x_1^2$, then $y_1 = (-x_1)^2$]. Hence the part of the curve for which $x$ is negative is the reflection in the $y$-axis of the part for which $x$ is positive. When this is the case, the curve is said to be *symmetrical about the $y$-axis*.

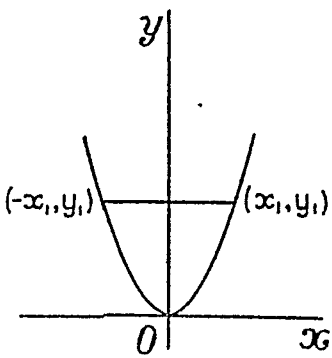



FIG. 2.17. Symmetry about the $y$-axis

Similarly if $(x_1, y_1)$ is a point on $y^2 = 4x$ so is $(x_1, -y_1)$ This curve is said to be *symmetrical about the x axis*

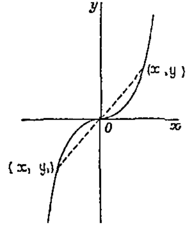


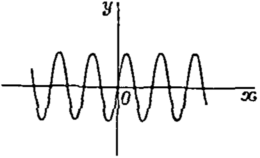

Fig 2 18 Symmetry about the origin

Another kind of symmetry is shown by $y = x^3$ If $(x_1\ y_1)$ is a point of this curve, so is $(-x_1, -y_1)$ [if $y_1 = x_1^3$, then $-y_1 = (-x_1)^3$] The middle point of $(x_1, y_1)$ and $(-x_1\ -y_1)$ is $(0, 0)$ the origin In this case the curve is said to be *symmetrical about the origin*

*Note* The gradient function also gives useful information when the graph is plotted from a table of values It enables us to decide how the points should be joined For example, the graph of $y = x^2+4x$ was completed by drawing a smooth curve through the plotted points This may be justified by noting that the gradient function increases steadily from large negative to large positive values If the curve 'zigzagged' between the plotted points, its gradient would fluctuate

Nevertheless care must be exercised in joining plotted points For example, the three points $(-1, -1)$, $(0, 0)$, $(1, 1)$, which lie on a straight line, are all on $y = x^3$ and yet its graph is not straight between $x = -1$ and $x = 1$ This is shown immediately by the gradient function $3x^2$, which is not constant

A more extreme example is the graph of $y = \sin(180x)^\circ$




$y = \sin(180x)^\circ$

Fig 2 19

From the following table of values a careless person might conclude that the graph coincides with the $x$ axis

| $x$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The next example is included to show how a curve can oscillate when it really tries. The equation of the curve is

$$y = \sin(36/x)^\circ.$$

Between $x = \frac{1}{10}$ and $x = \frac{1}{20}$ $y$ runs through all the values of $\sin\theta$ as $\theta$ increases from 0 to 360. This is repeated between $x = \frac{1}{20}$ and $x = \frac{1}{30}$, again between $x = \frac{1}{30}$ and $x = \frac{1}{40}$, and so on. Therefore between $x = 0{\cdot}1$ and $x = 0$, the curve describes more oscillations than can be counted and no pen can follow it.

| $x$ | 2 | 1 | 0·4 | 0·2 | 0·13 | 0·1 | 0·05 |
|---|---|---|---|---|---|---|---|
| $36/x$ | 18 | 36 | 90 | 180 | 270 | 360 | 720 |
| $y$ | 0·31 | 0·59 | 1 | 0 | $-1$ | 0 | 0 |

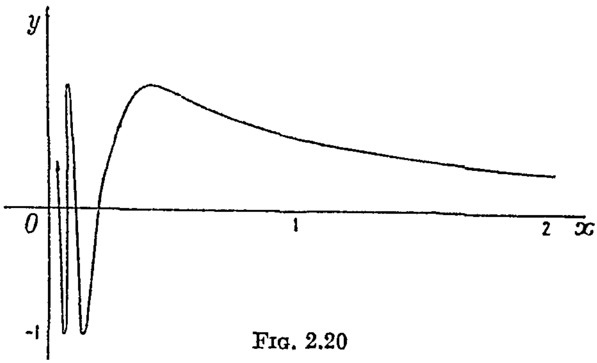



FIG. 2.20

EXERCISE 2.P

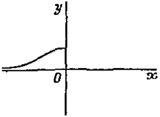
1–7. Sketch the following curves in the neighbourhood of the given point. $g$ denotes the gradient function:

1. $x = 1$, $y = 2$. Gradient at $x = 1$ is $\frac{1}{2}$ and is increasing.
2. $x = -1$, $y = 1$. Gradient at $x = -1$ is $-1$ and is increasing.
3. $x = 2$, $y = 4$. Gradient at $x = 2$ is $-2$ and is decreasing.
4. $x = -1$, $y = -1$. Gradient at $x = -1$ is 3 and is decreasing.
5. $x = 0$, $y = 1$, $g = x$. 6. $x = 0$, $y = 1$, $g = -x$.
7. $x = 0$, $y = 1$, $g = x^2$.

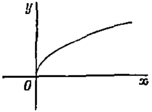
8–16. State whether the following curves have symmetry (*a*) about the $x$-axis, (*b*) about the $y$-axis, (*c*) about the origin:

8. $y = x^4+x^2+1$. 9. $y^2 = x$. 10. $y = x^5$.
11. $y^2+x^2 = 10$. 12. $x^2+y^2+2x = 0$. 13. $y = 1/x$.
14. $y = 1/x^2$. 15. $y = \sin x$. 16. $y = \cos x$.

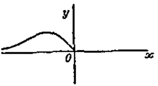
17–22 Use the stated property of symmetry to draw more of the curves of which parts are shown



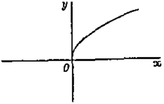

17 Symmetry about the $y$ axis



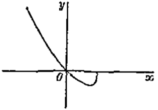

18 Symmetry about the $x$ axis



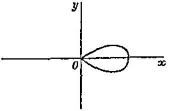

19 Symmetry about the origin




20 Symmetry about the $y$ axis




21 Symmetry about the $x$ axis




22 Symmetry about the origin

23–9 Sketch the curves whose equations are given

23 $y = x^2-2x$ 24 $y = 4x-x^3$ 25 $y = x^3$

26 $y = x^4$ 27 $y+x^3 = 0$ 28 $y+x^4 = 0$

29 $y^2 = 4x$

30 Draw the graph of $y = x^3-3x$ for values of $x$ from $-2$ to $2$

31 Draw the graph of $y = x^3+3x$ from $x = -2$ to $x = 2$

## 2 14 The graph of $y = 1/x$

Writing $g$ for the gradient function, we have $g = -1/x^2$ Here we meet a difficulty, the graph of the gradient function cannot be drawn any more easily than that of the given function However, we note that $x^2$ is always positive and therefore the gradient is always negative The curve falls for all values of $x$

Construct a small table of values:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $-10$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $10$ |
| $y$ | $-0{\cdot}1$ | $-0{\cdot}5$ | $-1$ | ? | $1$ | $0{\cdot}5$ | $0{\cdot}1$ |
| $g$ | $-0{\cdot}01$ | $-0{\cdot}25$ | $-1$ | ? | $-1$ | $-0{\cdot}25$ | $-0{\cdot}01$ |

Disregarding for the moment the entries marked ?, we can plot 6 points. The values of the gradient show how these may be joined.

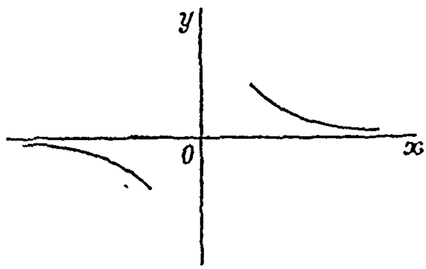



FIG. 2.21 (not to scale).

We need more information between $x = -1$ and $x = 1$.

| | | | | |
|---|---|---|---|---|
| $x$ | $-\frac{1}{2}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{2}$ |
| $y$ | $-2$ | $-10$ | $10$ | $2$ |
| $g$ | $-4$ | $-100$ | $-100$ | $-4$ |

Add these points to the sketch. Note the large negative values of $g$ when $x = \pm\frac{1}{10}$. The curve is plunging downwards very steeply at $x = -\frac{1}{10}$ and is coming down from the top edge of the paper equally steeply at $x = \frac{1}{10}$.

We now have two 'pieces of curve' and we probably feel instinctively that they should join up. But the curve cannot pass from $(-\frac{1}{10}, -10)$ to $(\frac{1}{10}, 10)$ on the paper without having a positive gradient. And we know that for all values of $x$ (except 0) the curve has a negative gradient. Consequently the two parts of the curve remain separate.

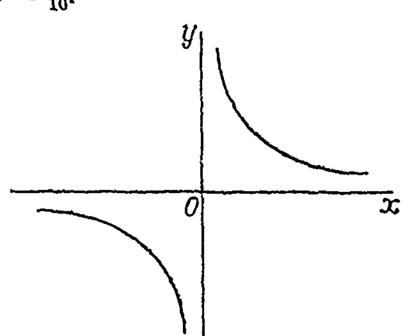



FIG. 2.22

This is a graph of a new kind. It is said to have two *branches*.

We must now consider the entries marked ? in the first table of values. If we put $x = 0$ in $y = 1/x$ we obtain $y = 1/0$. What

is the value of this ? Looking at the graph for a clue we imme diately encounter a difficulty One branch of the graph suggests that $1/0$ is a large positive number while the other branch suggests that it is a large negative number

Now suppose we buy a large sheet of paper and choose a small scale, for example, 1 in $= 1{,}000$ Then if the paper is long enough we may be able to plot $y$ coordinates as large as 1 million and $-1$ million For this the length of the paper must be 2 million/1,000 in $= 2{,}000$ in $= 166\frac{2}{3}$ ft , so it will have to be a *fairly large sheet Is it large enough to contain all the points of* the graph ? Suppose $x = 1/2$ million Then $y = 2$ million and this point cannot be plotted Similarly the point ($-1/2$ million, $-2$ million) cannot be plotted Also the gradient at each of these points is $-$(4 million million) and the two branches of the curve are separating even more violently than before

We must therefore, conclude that however large the sheet of paper is, the graph of $y = 1/x$ goes off the paper, as $x$ approaches 0 either through positive or negative values It is impossible to plot a point on the curve whose $x$ coordinate is 0

No number is large enough to represent $y$ in the formula $y = 1/x$ when $x = 0$ We say that $y = 1/x$ *does not exist when* $x = 0$ The operation of dividing by 0 is not defined in mathe matics and symbols like $1/0$, $1/0^2$ have no meaning

### Exercise 2 Q

1 Sketch the graph of $y = 1/x^2$

2 Represent the graphs of $y = x$ and $y = 1/x$ on the same figure by dotted lines Use these graphs to draw the graph of $y = x+1/x$ Note how the curve tries to approach $y = x$ when $x$ is large and $y = 1/x$ when $x$ is small

3 Give the equation of a graph which is near the graph of $y = x-1/x$, when (*a*) $x$ is large, (*b*) $x$ is small Sketch $y = x-1/x$

4 Sketch the graph of $y = x+1/x^2$ [First sketch $y = x$ and $y = 1/x^2$ ]

5 Sketch (*a*) $y^2 = x$, (*b*) $y^2 = 1/x$, (*c*) $y^2 = x+1/x$

### Exercise 2 R (in preparation for section 2 15)

Sketch the family of curves $y = x^2+c$ for $c = 0, 1, -1, 3$ fr $x = -3$ to $x = 3$

Find the gradient function of each curve

Draw the tangents to the curves at the points where $x = 2$

Find the value of $c$ for the curve of the family which pas through $(-1, -2)$

## 2.15. The equation of a curve which has a given gradient function

By this time you probably find it comparatively simple to write down the gradient function of a curve whose equation can be expressed as a sum of powers of $x$. But can we proceed in the reverse direction? Can we find the equation of the curve when we are given the gradient function?

For example, the gradient function of a curve is $2x$. What is the equation of the curve?

We solve this by trying to think of a function which has $2x$ as its gradient function. The first function we think of is $x^2$, but Ex. 2.R shows that $x^2+1$, $x^2-1$, $x^2+3$ are also answers to the problem. Generally we see that $y = x^2+c$ is a possible equation of the curve, where $c$ is any positive or negative number. Thus the answer to our problem is the equation of a family of curves and not the equation of a single curve. To obtain a definite curve further information must be given, e.g. the coordinates of a point through which the curve passes. Such problems are taken up in section 2.16.

We have seen that $y = x^2+c$ is a possible solution of the problem, but it is not yet clear that every solution is given by this formula with a suitable value of $c$. That this is in fact so, will now be proved.

Consider any two functions each of which has gradient function $2x$. Then the gradient function of their difference is $2x-2x = 0$ for all values of $x$. But if a function has zero gradient for all values of $x$ it is a constant (section 2.12 (4)). Therefore any two functions with gradient function $2x$ differ by a constant. But we have seen that one such function is $x^2$. Therefore all such functions are given by $x^2+c$, where $c$ is a constant.

### Exercise 2.S

Find the equation of the family of curves whose gradient function is:

1. $3x^2$. 2. $4x^3$. 3. $5x^4$. 4. 1.
5. $4x$. 6. $-2x$. 7. $1+2x$. 8. $3-2x$.
9. $x$. 10. $1+x$. 11. $2-x$. 12. $x/2$.
13. $-x/3$. 14. $2x/5$. 15. $1-4x/3$.
16. $(1-3x)/5$. 17. $2x^2$. 18. $x^2$.
19. $1+3x^2$. 20. $2x+x^2$. 21. $\frac{1}{2}x^2$.
22. $-x^2/4$. 23. $x^2/3$. 24. $3(x+x^2)$.
25. $(x-x^2)/5$. 26. $1-x+x^2$. 27. $x^3$.
28. $2x^3+3x^2$. 29. $(x^3+x)/5$. 30. $4x^3+6x^2$.
31. $\frac{1}{2}x^2-4x+2$. 32. $1/x^2$. 33. $-2/x^2$.

| | | |
|---|---|---|
| 34 $1/2x^2$ | 35 $x-1/x^2$ | 36 $-2/x^3$ |
| 37 $x^2+1/x^2$ | 38 $x^3-1/x^3$ | 39 $3x+1+2/x^2$ |
| 40 $4x-4/x^2$ | 41 $1/4x^3$ | 42 $5(x^2-2/x^2)$ |
| 43 $x-2/x^3$ | 44 $1/x^3-1/x^2$ | |

## 2.16

EXAMPLE 1

Find the equation of a curve with gradient function $2x$, passing through the point (2, 3)

*Solution* The equation of the family of curves with gradient function $2x$ is $y = x^2+c$ We now determine $c$ so that when $x = 2$, $y = 3$

We have $3 = 4+c$, giving $c = -1$

The curve is $y = x^2-1$

EXAMPLE 2 A body moves in a straight line so that $t$ sec after its start at $t = 0$, its velocity, $v$ f p s, is given by $v = 10-6t$ Find the distance time formula

*Solution* Let the distance from the start at $t$ sec be $s$ ft

Then $v$ is the gradient function of $s$

Hence $s = 10t-3t^2+c$, where $c$ is a constant

But $s = 0$ when $t = 0$ Therefore $c = 0$

Hence $s = 10t-3t^2$

This is the distance time formula

### EXERCISE 2 T

1–6 In the following the gradient function of a curve and the coordinates of a point on the curve are given Find the equation of the curve

| | | |
|---|---|---|
| 1 $2x$, (1, 2) | 2 $x$, (2, 5) | 3 $5+4x$, (1, 10) |
| 4 $4x$, (2, 8) | 5 $1-3x^2$, (0, 1) | 6 $4x^3$, (1, 0) |

7–16 Given the following velocity time formulae, find the corresponding distance time formulae assuming that $s$ is the distance from the start at $t = 0$

| | | |
|---|---|---|
| 7 $v = 1-2t$ | 8 $v = 4t-\frac{2}{3}$ | 9 $v = 20$ |
| 10 $v = 3t^2-5$ | 11 $v = 9(t^2-4)$ | 12 $v = 40+32t$ |
| 13 $v = 8t^3+1$ | 14 $v = 1+\frac{3}{4}t^2$ | 15 $v = 3t(2+t)$ |
| 16 $v = (1-2t)^2$ | | |

17–19 We are not always given that $s = 0$ when $t = 0$ These are examples of different initial conditions

17 $v = 4t$ and $s = 5$ when $t = 1$ Find $s$ when $t = 2$ Find $s$ when $t = 0$

18 $v = 8t-4$ and $s = 9$ when $t = \frac{1}{2}$ Find $s$ when $t = 1$

19. $v = 3t^2+2t$ and $s = 22$ when $t = 1$. Find $s$ when $t = 4$. What are the values of $s$ and $v$ when $t = 0$?

20–4. Find the equations of the curves satisfying the given conditions:

20. Gradient function $1\frac{1}{2}$ and passing through (2, 0).

21. Gradient function $x^2$ and passing through $(1, \frac{4}{3})$.

22. Gradient function $\frac{1}{2}x$ and passing through $(1, \frac{1}{4})$.

23. Gradient function $1/x^2$ and passing through $(\frac{1}{5}, 5)$.

24. Gradient function $(1-x)$ and passing through (2, 4).

(25–30. *These refer to the motion of a body in a straight line starting at* $t = 0$.)

25. If $v = t/10$ f.p.s. find the distance travelled in 10 sec. from the start.

26. If $v = (1+3t^2)$ yd. per min. find the distance travelled from $t = 1$ to $t = 2$ (i.e. during the second minute).

27. If $v = (2+\frac{1}{2}t^3)$ yd. per min., find the distance travelled during the second minute of the motion.

28. If $v = (4t-1)$ f.p.s., find the distance travelled during the fourth second of the motion.

29. If $v = 6t$ f.p.s., find the distance travelled while the velocity increases from 6 to 12 f.p.s.

30. If $v = (1+t)$ cm.p.s., find the distance in which the velocity is increased from 5 cm.p.s. to 11 cm.p.s.

31. The gradient function of a family of curves is $\frac{1}{2}(1+x)$. Show that each curve of the family rises by the same amount as $x$ increases from 2 to 4. What is the rise?

32. The gradient function of a curve is $3-4x$ and it passes through (0, 2). Find the equation of the curve and the points in which it meets the axes. Draw a sketch. Find the decrease in $y$ as $x$ increases from 1 to 5.

33–52. Find the equations of the families of curves with the given gradient functions:

| | | | |
|---|---|---|---|
| 33. $1-\frac{1}{3}x$. | 34. $6x^2$. | 35. $6/x^2$. | 36. $5x^3$. |
| 37. $-4/x^3$. | 38. $1/8x^3$. | 39. $\frac{1}{2}$. | 40. $\frac{5}{2}x$. |
| 41. $\frac{3}{2}-\frac{2}{3}x$. | 42. $9-10x^2/3$. | 43. $2x-1/2x^2$. | |
| 44. $x^2/3-3/x^2$. | 45. $5-4x+6x^2$. | 46. $x^2-2+1/x^2$. | |
| 47. $(x-1)(x-2)$. | 48. $(x+1)^2$. | 49. $(1-x)(x-4)$. | |
| 50. $(1+x^2)/x^2$. | 51. $4(2-x^2)/x^2$. | 52. $\frac{3}{2}x(1+x)^2$. | |

53. The gradient function of a curve is $6x(x-1)$ and it makes an intercept of 2 on the $y$-axis.

Find the equation of the curve.

54 A body travels in a straight line and its velocity, $v$ yd per min at $t$ min is given by $v = (3t-2)(t+4)$ Find the distance travelled during the fifth minute of the motion

55 If a body moves in a straight line and its velocity at $t$ sec is $v = \frac{1}{5}(7+3t^2)$ f p s, find the distance in which the velocity is increased from 2 f p s to 11 f p s

## Miscellaneous Exercise 2.X

1 What is the gradient function of $y = x^3-3x$? Find the gradient when $x = 2$ and the average gradient from $x = 0$ to $x = 2$

2 At the point $(-1, 2)$ the gradient of a curve is 2 and in the neighbourhood of this point the gradient increases as $x$ increases Draw a sketch of the curve in the neighbourhood of $(-1, 2)$

3 Find the equation of the tangent at $x = -2$ to $y = x(4-x^2)$

4 The gradient function of a curve is $x^2-2x$ and it makes an intercept of 1 on the $y$ axis Find the equation of the curve

5 Find the gradient function of $y = (2-x)(1+x)$ Sketch the curve

6 $y = (1-x^3)/x^4$ What is the rate of change of $y$ with $x$ when $x = 2$?

7 If $s = 3t+t^3$ (units ft, min) is the distance time formula for a body moving in a straight line, find the velocity at $t = 0$ and $t = 10$ Also find the average velocity during this 10 min

8 Sketch the curves (a) $y = -2x^2$, (b) $x = -4y^2$

9 Give the gradient function of $y = (x^4-2x^2+4)/x^3$

10 If the rate of change of $v$ with $h$ is $\pi h^2(2-h)$ and $v = \frac{1}{2}\pi$ when $h = 1$, find $v$ in terms of $h$

---

11 Write down the gradient function of $y = (x-1)^2$ Sketch the curve

12 At $(-2, 0)$ the gradient of a curve is $-1$ and is increasing Sketch the curve near $(-2, 0)$

13 About which axis is $y = 1+1/x^2$ symmetrical? Sketch the curve

14 Find the points of contact of the tangents to $y = x+3/x$ which are parallel to $y+2x = 0$

15 The rate of change of $y$ with $x$ is $(3x-1)(x-1)$ and when $x = 3$ $y = 8$ Find the formula

16 Find the gradient function of $x^5+1/x^5$

17 Without using the rule, work out the gradient function of $2x^2$

18 Find the equations of the tangents at the three points where $y = x(x^2-1)$ crosses the $x$ axis

19. Write down the equation of the family of curves whose gradient function is $(x^4-1)/x^3$.

20. The height of a balloon above the ground in ft. at $t$ sec. is given by $h = 1000-20/t$ when $t \geqslant 1$. Find a formula for the vertical velocity of the balloon. Is it rising or falling? Find its height and velocity at $t = 5$.

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21. Find the points of contact of the tangents to $6y = 2x^3+3x^2+24x$ which are perpendicular to $6y+x = 4$.

22. Give the gradient function of $x(\pi-2x^3)$.

23. Without using a rule, find the gradient function of $y = x+x^2$.

24. A body moves in a straight line so that $v = (t^3-2t)$ f.p.s. Find the distance travelled from $t = 1$ to $t = 4$.

25. Find the rate of change of $y$ with $x$ when $y = x^3-x^2$ and (*a*) $x = 1$, (*b*) $x = 2$, (*c*) $x = 0$. For what values of $x$ is the rate of change of $y$ with $x = 40$?

26. Sketch $y = 1/x$. Find the intercepts on the axes made by the tangent at $x = 2$.

27. $y$ is given in terms of $x$ by the formula $3y = x^3-3x^2+3x$. Show that as $x$ increases $y$ always increases.

28. Give the gradient function of $x(2+x)^2$.

29. The line $y = 2x+1$ touches a curve whose gradient function is $4x$. Find the coordinates of the point of contact and the equation of the curve.

30. A road has the form of $y = x^3$ from $x = -2$ to $x = 1$ and is straight from $x = 1$ to $x = 4$. (There is no corner at $x = 1$ so that the straight portion is a tangent to the curved portion at $x = 1$.) Sketch the plan of the road. What is the angle, to the nearest degree, between the directions of motion of a car at $x = -2$ and $x = 4$ if it is travelling from the first point to the second?

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31. If $v = (12-3t)$ f.p.s. and the body starts at $t = 0$, find the time and the distance from the starting-point at which the body is instantaneously at rest.

32. The rate of change of $y$ with $x$ is $kx$ where $k$ is a constant. When $x = 1$, $y = 9$ and when $x = 5$, $y = 57$. Determine $k$ and the formula for $y$ in terms of $x$.

33. The gradient function of a family of curves is $x-1/x^2$. Find the equation of the family. One of the curves passes through the point (1, 2). What is the equation of the tangent to this curve at (1, 2)?

34. Take the origin at the top of a hill with the $x$-axis horizontal and the $y$-axis vertically upwards. The gradient of the hill-side at a

horizontal distance $x$ ft from the origin is $-x/1000$ What is the depth of the ground below the hill top when (a) $x = 500$, (b) $x = 1000$?

35 Find the gradient functions of (a) $(x+1)(x+2)$, (b) $x(x+1)(x+2)$

36 The surface area, $A$ sq in, of a lidless cylindrical tin of radius $r$ in and height 10 in is given by $A = \pi r(r+20)$ Show that the rate of change of $A$ with $r$ is never less than $20\pi$, and find its value when $r = 5$

37 Find the coordinates of the points of intersection of $y = 3x^2+5x+1$ and $y = 3$ Find the equations of the tangents to the curve at these points

38 Find the curve whose gradient function is $x^{-4}$ and which passes through $(-\frac{1}{3}, 10)$

39 The distance time formula for a body is $s = t-t^2/10$ (units ft, sec) Find the velocity of the body 4 sec after the start When is the body travelling four times as fast (a) in the same direction, (b) in the opposite direction?

40 Find the equation of the tangent at the point where $x = \frac{2}{3}$ on $y = 3x^3-\frac{2}{9}$ Find the equation of the other tangent to the curve which is parallel to this one

---

41 The direction angle, $\alpha$, of the tangent at any point $(x, y)$ of a curve is given by $\tan\alpha = 1-2x^2$ Find the equation of the curve if it passes through the point $(-1, 1\frac{2}{3})$

42 Find the gradient function of $x^2(1-2x)+x(1+2x)$

43 If $v = t^2-t$, find the distance in which the body increases its velocity from 0 to 20

44 Sketch the curves (a) $y = x^2-4x$, (b) $y = 2x-x^2$

45 A curve has the gradient function $6x^2-5$ and touches $y-x+3 = 0$ where $x = -1$ Find its equation

46 Find the equations of the tangents to $8y = x^2(x-5)$ at the points where $x = 2$ and $x = 5$

47 If $g = -10/x^3$ (where $g$ denotes the gradient function) and $y = 10$ when $x = \frac{1}{2}$, find $y$ when $x = 2\frac{1}{2}$

48 A body starts from the origin and moves along the $x$ axis so that after $t$ sec its position is given by $x = t^2(t-6)$ Find $x$ when $v$ is again 0 Also find $v$ when $x$ is again 0

49 Two quantities are connected by the formula $pv = 100$ Find the rate of change of $p$ with $v$ when $v = 10$ Also find the rate of change of $v$ with $p$ when $v = 20$

50 Show that there are two values of $a$ such that the tangents at $x = 1$ and $x = 3$ to the curve $y = ax^2-2x-1$ are perpendicular, and find these values

## Miscellaneous Exercise 2.Y

1–10. Find the gradient functions of:

1. $6x^3$.
2. $\frac{1}{2}(1+x)^2$.
3. $x^2-2/x$.
4. $(x^3+1)/x^2$.
5. $(x+2)^2/2x$.
6. $x^{10}+1+1/x^{10}$.
7. $10\pi x^2(2-x)$.
8. $4\pi x^2(1+x^2)$.
9. $(x+k)^2$ [$k$ is a constant].
10. $x(x+1)(x+4)$.

11–20. Find the functions which have the following gradient functions:

11. $2x^3$.
12. $\frac{1}{8}x^4$.
13. $1+2x+3x^2$.
14. $4\pi x^2$.
15. $(1+3x)^2$.
16. $1-1/x^2$.
17. $3(x^2-4/x^2)$.
18. $4x^3+1+4/x^3$.
19. $(x+k)^2$ where $k$ is a constant.
20. $3(5-x)/x^3$.

21–2. Given the following distance-time formulae find the corresponding velocity-time formulae:

21. $s = 5t^4+t^2-10$.
22. $4s = (t-3)(t-5)$.

23–4. Given the following velocity-time formulae find the corresponding distance-time formulae:

23. $v = 5(t^2-6)^2$ and $s = 0$ when $t = 0$.
24. $v = (t^2+1/t)^2(t > 1)$ and $s = 0$ when $t = 1$.

25. Find the equation of the curve passing through the origin for which the gradient function is $x^2-3$.

26. Find the equations of the tangents to $y = x+1/x$ which are parallel to the $x$-axis.

27. Find the equation of the tangent to $y = 4/x^2$ whose direction angle is (*a*) 45°, (*b*) 135°. Show that these tangents meet on $x = 0$.

28. The gradient function of a curve is $4+2/x^2$ and it passes through (2, 3). Find the equations of the two tangents whose gradient is 6.

(*Nos.* 29–34 *refer to the movement of a body in a straight line.*)

29. If the distance-time formula is $s = t+t^2$ (units: ft., sec.), find the average velocity (*a*) from 1 to $(1+h)$ sec.; (*b*) from 1 to 1·01 sec.; (*c*) from 1 to 1·001 sec.

What is the velocity at 1 sec.?

30. $s = 48t-t^3$ (units: ft., sec.), and the motion starts at $t = 0$.
   (i) When is the velocity zero?
   (ii) When is the body again at the starting-point?
   (iii) What is the distance from the starting-point (*a*) when $t = 2$, (*b*) when $t = 5$?

(iv) How far has the body actually travelled (a) when $t = 2$, (b) when $t = 5$?
(v) How far does the body travel during the fourth second of its motion?
(vi) What is its average velocity during the first four seconds?
(vii) What is the velocity when $t = 3$?
(viii) Describe the motion

31 A stone is projected from a cliff so that it falls vertically to the sea After $t$ sec its height above the sea, $h$ ft, is given by

$$h = 144+128t-16t^2$$

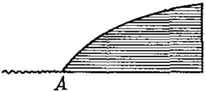
Find
(a) The velocity of projection Is it upwards or downwards?
(b) The greatest height of the stone above the sea (Consider what the velocity must be when the stone is at its highest point)
(c) The time and velocity when it strikes the sea
(d) The height of the cliff

32 If $v = (2t-10)$ f p s and $s = 0$ when $t = 0$, find when the body is at rest and the distance from the starting point, $s = 0$, at that time Find the velocity with which it passes through the starting point again and describe the motion as completely as you can

33 If $v = (1-t^2/3)$ f p s and the body starts from $s = 0$ at $t = 0$,
(a) Find the distance from the starting point after $t$ sec
(b) Find when the body returns to the starting point and its velocity as it passes through
(c) Show that when the body changes its direction of motion, the distance from the starting point is $\frac{1}{3}(2\sqrt{3})$ ft

34 $v = (12t^2-30t+12)$ f p s and the motion starts at $t = 0$ Find the distances of the body from the starting point at the two instants when the velocity is zero

35 The sketch shows half the cross section of an artificial bank built at the side of a river $A$ is a point at the bottom of the bank, and at a horizontal distance of $x$ ft from $A$ the gradient of the cross section is $1-x/10$ At what horizontal distance from $A$ is the gradient zero? If this is the top of the bank, find the height of the top above the level of $A$




36 The following table gives the times (Greenwich Mean Time) of sunrise and sunset in latitude 50° N on Sundays in January 1944

| *Date* | *Time of sunrise* | *Time of sunset* |
|---|---|---|
| 2 Jan. | 07 59 | 16 09 |
| 9 Jan. | 07 57 | 16 17 |
| 16 Jan. | 07 53 | 16 26 |
| 23 Jan. | 07 47 | 16 37 |
| 30 Jan. | 07 38 | 16 49 |

(i) Find in min. per week, to the nearest min., the rates of change of the times of sunrise and sunset (*a*) on 9 Jan., (*b*) on 23 Jan. (Suggested scales for the graphs: 1 in. = 10 min.; 0·7 in. = 1 week.)

(ii) Find also the rate of change of the length of the day on 9 Jan. and on 23 Jan. to the nearest min. per week.

(iii) Find the average rate of change of the times of sunrise and sunset during the month (take it from 2 to 30 Jan.) in min. per week.

37. The following table gives the altitude of the sun at various times on a certain day:

| | | | | |
|---|---|---|---|---|
| Altitude | 10° 33′ | 11° 58′ | 13° 22′ | 14° 47′ |
| Time (hr. min.) | 06 19 | 06 27 | 06 35 | 06 43 |

| | | | | | | |
|---|---|---|---|---|---|---|
| Altitude | 56° 54′ | 56° 58′ | 56° 59′ | 56° 57′ | 56° 54′ | 56° 47′ |
| Time | 11 55 | 12 01 | 12 04 | 12 15 | 12 19 | 12 25 |

(*a*) Show that the rate of change of altitude is approximately constant from 06 19 to 06 43 and find its value.

(*b*) Find the rates of change of altitude at 12 00 and 12 20.

(*c*) Find, as accurately as you can, the maximum altitude of the sun and the time at which it occurs.

38. The following table shows the times taken by the former L.N.E.R. locomotive No. 2002 *Earl Marischal* to travel certain distances at the start of an actual run from Aberdeen to Dundee.†

| *Distance from Aberdeen (miles)* | *Time from start (min. sec.)* |
|---|---|
| 0·0 | 0 00 |
| 0·6 | 2 19 |
| 1·1 | 3 28 |
| 3·1 | 6 55 |
| 4·8 | 9 40 |

Find, approximately, the speed, in m.p.h., at 2 min.

Show that the speed becomes approximately constant and find its value.

† From *Locomotives of Sir Nigel Gresley*, by O. S. Nock, p. 124.

39 A ball is let fall from rest, 15 ft above the floor of a hall, and its heights at various times are given in the following table The ball hits the floor and rebounds after 1 sec

| Height (ft ) | 15 | 14 4 | 12 6 | 9 6 | 5 4 | 0 | 3 4 | 5 6 | 6 6 |
|---|---|---|---|---|---|---|---|---|---|
| Time (sec ) | 0 | 0 2 | 0 4 | 0 6 | 0 8 | 1 0 | 1 2 | 1 4 | 1 6 |

(a) Find the velocity when $t = 0\ 5$ and $1\ 3$, stating the direction of motion

(b) Find, as accurately as you can, the velocity with which the ball hits the floor and the velocity of rebound

40 $y = x^3-4x^2+2x$ Find the value of $x$ at which the rate of change of $y$ with $x$ has the same value as when $x = 3$

41 $y = x^3-3x^2+4$ Find a formula in which the rate of increase of the dependent variable ($z$) with the independent variable ($x$) is $x$ times the rate of increase of $y$ with $x$ for every value of $x$, and $y = z$ when $x = 0$

42 The length of the mandible, $y$ mm, of a stag beetle whose total length is $x$ mm is given approximately by $y = -5\ 64+0\ 368x$ † Show that the rate of increase of the length of the mandible with the length of the body is constant and find its value

The corresponding relation for the reindeer beetle is given as $x = 1\ 7y+13\ 7$ Find the rate of increase of the mandible ($y$) with the body length ($x$) for this beetle and compare with that for the stag beetle

43 Find the coordinates of $A$ and $B$, the points where $y = 2$ meets $y = 2x^2-x+1$ Find the equations of the tangents at $A$ and $B$ and the equation of the line joining the middle point of $AB$ to the point of intersection of these tangents

44 Find the equations of the tangents at the points on the curve $y = x^3-6x^2+6x$ where $x = 1$ and $x = 2$ and show that they meet on the $x$ axis

45 Find the equations of the tangents to $y = x^3-5x^2+9x-1$ at the points where $x = 1$ and $x = 3$, and show that the first tangent passes through the point of contact of the second

46 Find the equation of the tangent to $6y = x^2(x-1)$ at the point where $x = -1$ Verify that this tangent meets the curve again at the point (3 3) Find the equation of the tangent at (3, 3)

47 Find the coordinates of the points of intersection of $y = x^2(4-x)$ and $y = 4-x$ Find the equations of the tangents at these points and the coordinates of the vertices of the triangle of which they are the sides

† D'Arcy Thompson *Growth and Form*, p 209

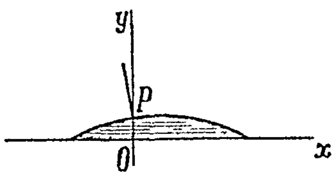
48. Find the gradient at each of the points where

$$y = ax^2(x-1)(x-3)$$

meets the $x$-axis, $a$ being a constant. Find the only positive value of $a$ which makes two of these tangents perpendicular. When $a$ has this value find the equations of the tangents and the area of the triangle formed by them.

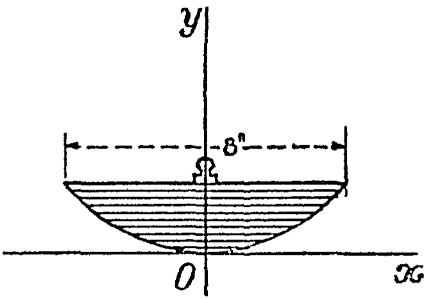
49. Find the equations of the tangents to $y = \frac{1}{2}(x+1/x)$ which are parallel to $2y+3x = 0$.

50. The cross-section of a hill is $10y = 3+x-x^2$ (where the unit of $x$ and $y$ is 100 yd.). At $P \equiv (0, 0{\cdot}3)$ a flagstaff is set up at right angles to the ground. What angle does it make with the vertical (i.e. $Oy$) ?




51. The figure shows a desk blotter with a curved boundary given by $8y = x^2$, the unit of $x$ and $y$ being 1 in.

Through what angle has it turned from the position shown, when a point on the curved boundary 2 in. from the line of symmetry ($Oy$) is in contact with the desk ($Ox$) ?




52. Complete the following table for the curve $y = x^3(x^2-5)$. ($g$ denotes the gradient function.)

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | | | | | |
| $g$ | | | | | |

Plot these points; they lie on a straight line.

Is the graph straight from $x = -2$ to $x = 2$ ?

Show that it is symmetrical about the origin and that its gradient is zero when $x = \sqrt{3}$. Add this value of $x$ and, by symmetry, another value to the table of values. Draw the curve from $x = -2$ to $x = 2$.

## III

## NOTATION AND APPLICATIONS

### 3 1. Increments

If $y$ and $x$ are connected by a formula we defined the average rate of change of $y$ with $x$ in section 2 7 as

$$\frac{y_2-y_1}{x_2-x_1} \quad \text{or} \quad \frac{\text{change of } y}{\text{change of } x}$$

The change of either variable is called the *increment* of that variable If $x$ denotes the independent variable, $(x_2-x_1)$ is the increment of $x$ and $(y_2-y_1)$ is the increment of the dependent variable, $y$, resulting from the given increment of $x$ If $(x_2-x_1)$ is positive, $x_2$ is greater than $x_1$ and $x$ is increasing, while if $(x_2-x_1)$ is negative $x$ is decreasing Similarly $(y_2-y_1)$ is positive or negative according as $y$ increases or decreases Thus the increments are directed quantities A positive increment defines an increase of the variable, a negative increment defines a decrease of the variable

A special notation has been invented to represent increments An increment of $x$, $x_2-x_1$, is represented by $\delta x$ or $\Delta x$ [$\Delta$, $\delta$ = capital and small delta, Greek $d$, the first letter of the word 'difference', the increment being the difference (new value-old value) of the variable ]

The increment of $y$, the dependent variable, caused by the increment, $\delta x$, of $x$ is represented by $\delta y$ [if the increment of $x$ is denoted by $\Delta x$, the increment of $y$ is denoted by $\Delta y$]

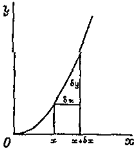


Fig 3 1

If $y = x^2$ and $x$ changes from 1 to 3, we write

$$x = 1, \quad x+\delta x = 3, \quad \delta x = 2,$$

and

$$y = 1, \quad y+\delta y = 9, \quad \delta y = 8$$

If $x$ changes from 4 to 2 we have

$$x = 4, \quad x+\delta x = 2, \quad \delta x = -2,$$

and

$$y = 16, \quad y+\delta y = 4, \quad \delta y = -12$$

More generally, if $y = x^2$ and $x$ changes to $x+\delta x$ we have

$$y = x^2$$

and $$y+\delta y = (x+\delta x)^2 = x^2+2x.\delta x+(\delta x)^2.$$

Hence by subtraction

$$\delta y = 2x.\delta x+(\delta x)^2. \tag{1}$$

In section 2.9, where we found the gradient function of $x^2$, we wrote $$`y_1 = x_1^2, \qquad y_2 = (x_1+h)^2,$$

therefore $$y_2-y_1 = 2x_1 h+h^2.' \tag{2}$$

(1) conveys the same meaning as (2) but more concisely. Note that $\delta x$ replaces $h$, a single letter. $\delta x$ must always be thought of as a single quantity and care must be taken not to confuse the delta notation with the algebraic notation where, for example, $ab$ denotes the product $a \times b$.

One advantage of the delta notation is that it distinguishes clearly the increments of $x$ and $y$, e.g. $\delta x$ is obviously the increment of $x$ and not of $y$. Another advantage is that, once the notation has been defined, we understand without explanation that $\delta y$ is the increment in the dependent variable *caused* by the increment $\delta x$ in the independent variable.

The notation may be applied equally well to a formula connecting two variables represented by letters other than $x$ and $y$. For example, the distance, $s$ ft., fallen by a stone in $t$ sec. is given by $s = 16t^2$. If we let $t$ increase from $t$ to $t+\delta t$ we use $\delta s$ to represent the resulting increment in $s$, i.e. the distance fallen in that time.

We have $$s = 16t^2$$

and $$s+\delta s = 16(t+\delta t)^2 = 16t^2+32t.\delta t+16(\delta t)^2.$$

Therefore $$\delta s = 32t.\delta t+16(\delta t)^2.$$

Let $t = 2$ and $\delta t = \frac{1}{2}$. Then the distance fallen from 2 to $2\frac{1}{2}$ sec. is given by

$$\begin{aligned} \delta s &= 32.2.\tfrac{1}{2}+16(\tfrac{1}{2})^2 \\ &= 32+4 \\ &= 36 \text{ ft.} \end{aligned}$$

### Exercise 3.A

1. If $y = x^2$, read off the values of $\delta y$, from a table of squares when (*a*) $\delta x = 1$, $x = 52$; (*b*) $\delta x = 2$, $x = 20$.

2. If $y = \sin x$, find $\Delta y$ approximately from the tables when $x = 30°$, $\Delta x = \frac{1}{20}°$.

3. If $y = \log x$, read off an approximation to $\delta y$ when $x = 370$, $\delta x = 0{\cdot}1$.

4 If $y = 1/x$ and, (a) $x = 4$ $\Delta x = 0\ 7$, (b) $x = 6$ $\Delta x = 0\ 3$ find an approximation to $\Delta y$ in each case, from a table of reciprocals

5 Use the following data to write down the value of $\sqrt{33\ 2}$ approximately If $y = \sqrt{x}$, $x = 33$, and $\delta x = 0\ 2$, then $y = 5\ 745$ and $\delta y = 0\ 017$

6 Use the following data to write down $1/5\ 2$ approximately If $y = 1/x$, $x = 5$, and $\delta x = 0\ 2$ then $y = 0\ 2$ and $\delta y = -0\ 0077$

7 When $y = \sqrt{x}$, $x = 100$, and $\delta x = -2$, then $\delta y = -0\ 101$ Find $\sqrt{98}$

8 Use $30^2 = 900$ and $\delta y = 2x\ \delta x + (\delta x)^2$ to calculate the squares of 31, 32, 33, and 34 Check your results from a table of squares

9 If the distance $s$ travelled by a body in time $t$ is given by $s = 2t^2$ find $\Delta s$ when $t = 1$ and $\Delta t = \frac{1}{2}$

10 If the velocity of a body at time $t$ is given by $v = \frac{1}{4}t^2$, find $\delta v$ when $t = 2$ and $\delta t = 1$

11 If $y = x + x^2$, find $\delta y$ in terms of $x$ and $\delta x$

12 If $y = 1 - 3x^2$, find $\delta y$ in terms of $x$ and $\delta x$

13 If $r$ and $s$ are connected by the formula $r = 100/s$, find $\Delta r$ in terms of $s$ and $\Delta s$ Also find $\Delta r$ when $s = 10$ and $\Delta s = 0\ 1$

14 If $n$ is a number, its reciprocal, $r$, is given by $r = 1/n$ Find $\delta r$ in terms of $n$ and $\delta n$

Calculate $\delta r$ when (a) $n = 1$, $\delta n = \frac{1}{2}$, (b) $n = 10$, $\delta n = -1$

15 Draw the graph of $y$ against $x$,

(a) if $\delta y = 2\delta x$, and $y = 2$ when $x = 0$,

(b) if $\delta y = -\delta x$, and $y = 1$ when $x = 0$

16 How can we use the delta notation to represent the average rate of change of $y$ with $x$?

## 3 2 Differentials

$P \equiv (x\ y)$, $Q \equiv (x+\delta x\ \ y+\delta y)$ are two points on the curve shown in Fig 3 2 where $PN$ represents $\delta x$ and $QN$ represents $\delta y$ Then

$$QN/PN = \delta y/\delta x$$

is the average gradient of the curve from $x$ to $x+\delta x$

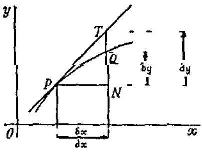


Fig 3 2

The gradient of the curve at $P$ is the gradient of the tangent at $P = TN/PN$

$TN$ is called the *differential* of $y$ and is written $dy$ Since $PN$ is $\delta x$, the gradient at $P$ may now be written $dy/\delta x$ This looks

awkward and we write $dx$ for $\delta x$ so that the gradient at $P$ is $dy/dx$. $dx \equiv \delta x$ and each is the increment of $x$. No confusion is caused in practice by having two names for the same quantity. If we are measuring the 'rise' to the curve, $\delta y$, we use the symbol $\delta x$ for the increment of $x$, while if we are measuring the 'rise' to the tangent we use $dx$. For example, let the curve be $y = x^2$. (Fig. 3.3.)

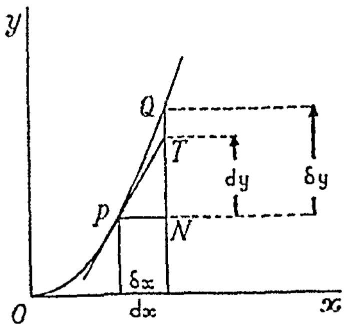



Fig. 3.3

Let $P \equiv (x, y)$

and $Q \equiv (x+\delta x, y+\delta y)$.

Then $\delta y = (x+\delta x)^2 - x^2 = 2x.\delta x + (\delta x)^2$.

The gradient at $P = 2x$.

Therefore $$\frac{TN}{PN} = 2x$$

or $$\frac{dy}{dx} = 2x$$

or $$dy = 2x\,dx.$$

Put $x = 3$ and $\delta x = dx = 0{\cdot}1$.

Then $\delta y = 6 \times 0{\cdot}1 + 0{\cdot}01 = 0{\cdot}61$

and $dy = 6 \times 0{\cdot}1 = 0{\cdot}6$.

Put $x = 3$ and $\delta x = dx = 0{\cdot}05$.

Then $\delta y = 6 \times 0{\cdot}05 + 0{\cdot}0025 = 0{\cdot}3025$

and $dy = 6 \times 0{\cdot}05 = 0{\cdot}3$.

## Exercise 3.B

1. If $y = 4x^2$, write down the gradient function. Find $dy$ in terms of $x$ and $dx$.

2–6. Find $dy$ in terms of $x$ and $dx$ for the following curves. Show $dy$ on a sketch in Nos. 2 and 3.

2. $y = x^3$. 3. $y = 1/x$. 4. $y = 2x - 8x^2$.

5. $y = x^2 + \frac{1}{6}x^3$. 6. $y = 1 + \frac{1}{2}x^4$.

7. Find $\delta y$ and $dy$ at the point (2, 8) on the curve $y = 2x^2$ in terms of $\delta x$ or $dx$. Draw a sketch and mark $\delta y$ and $dy$ on it. Find $\delta y$ and $dy$ when $\delta x = 1$.

8. Find $\delta y$ and $dy$ at the point (1, 1) on $y = 1/x$ for any increment of $x$. Find $\delta y$ and $dy$ when $\delta x = \frac{1}{10}$.

9. Find (a) $\delta y$, (b) $dy$, (c) $\delta y - dy$ at the point where $x = 1$ on $y = 10x^2$ for any increment of $x$. Show on a sketch the geometrical meaning of $\delta y - dy$. Calculate $\delta y - dy$ when $\delta x =$ (i) $0{\cdot}1$, (ii) $0{\cdot}01$.

10. A curve passes through the point (2, 0) and when $x = 2$, $dy = 2dx$. The values of $\delta y$, for $x = 2$ and various values of $\delta x$, are:

| | | | | | |
|---|---|---|---|---|---|
| $\delta y$ | 0 21 | 0 44 | 0 69 | 0 96 | 1 25 |
| $\delta x$ | 0 1 | 0 2 | 0 3 | 0 4 | 0 5 |

Draw the curve from $x = 2$ to $x = 2\ 5$.

Plot points on the tangent when $dx = 0{\cdot}2$ and 0 5.

Draw the tangent.

11. A curve passes through the origin and when $x = 0$, $dy = 3dx$. Values of $\delta y$, for $x = 0$ and various values of $\delta x$, are:

| | | | | |
|---|---|---|---|---|
| $\delta y$ | 0 75 | 0 63 | 0 | $-2$ 25 |
| $\delta x$ | 0 5 | 0 7 | 1 | $-0$ 5 |

Sketch the curve from $x = -0{\cdot}5$ to $x = 1$ and its tangent at the origin.

12. *Complete the following table for the curve* $y = x^2$ *at* $x = 4$

| | | | |
|---|---|---|---|
| $x = 4.\ \ \delta x$ | 0 2 | 0 1 | 0 01 |
| $\delta y$ | | | |
| $dy$ | | | |
| $\delta y - dy$ | | | |

13. At (1, 5) on a certain curve $dy = 10dx$. Complete the following table·

| | | | | |
|---|---|---|---|---|
| $x = 1\ \ \delta x$ | 0 5 | 0 3 | 0 1 | 0 01 |
| $\delta y$ | 6 25 | 3 45 | 1 05 | 0 1005 |
| $dy$ | | | | |
| $\delta y - dy$ | | | | |

Draw the curve between $x = 1$ and $x = 1{\cdot}5$ and its tangent at $x = 1$.

14. At $(-2, 0)$ on a certain curve $dy = -2dx$. Complete the following table:

| | | | | |
|---|---|---|---|---|
| $x = -2.\ \ \delta x$ | 0 1 | 0 5 | 1 | 2 |
| $\delta y$ | $-0$ 19 | $-0$ 75 | $-1$ | 0 |
| $dy$ | | | | |
| $\delta y - dy$ | | | | |

Draw the curve between $x = -2$ and $x = 0$ and its tangent at $x = -2$.

15. A curve passes through the origin and when $x = 0$, $\delta y = \delta x - (\delta x)^2$ and $dy = dx$. Find $\delta y - dy$ and sketch the curve, with its tangent at the origin, from $x = -1$ to $x = 1$.

[Note the information given by the sign of $\delta y - dy$ about the curve and its tangent in Nos. 13, 14, 15 ]

16. Draw the curve $y = x^2-2x-1$ from the table

| | | | | | |
|---|---|---|---|---|---|
| $y$ | 2 | $-1$ | $-2$ | $-1$ | 2 |
| $x$ | $-1$ | 0 | 1 | 2 | 3 |

Write down $dy$ in terms of $x$ and $dx$. Draw the tangent at $x = 2$ by finding $dy$ when $x = 2$ and $dx = 5$ and drawing a straight line between two points. Without drawing any more of the curve, calculate the distance between the curve and its tangent at $x = 2$, measured along the line $x = 7$.

Draw the tangents to the curve at $(0, -1)$ and $(1, -2)$ by similar constructions.

17. Show that on the curve $y = x^2$, $\delta y - dy$ is always positive for any increment of $x$, positive or negative. Show what this means by a sketch of the curve in the neighbourhood of a point $P$ together with the tangent at $P$.

18–21. Find $dy$ in terms of $x$ and $dx$ when:

18. $y = x(x+1)^2$. 19. $y = (x-2)^2/x$.

20. $y = x^3-4x^2+10x-6$. 21. $y = x^2-2/x^2$.

## 3.3. The application of differentials to formulae

The air resistance, $R$ lb. wt., on a certain motor-car travelling at $v$ m.p.h. is given by the formula $R = v^2/10$. Suppose we want to find the increase in the air resistance when $v$ increases from 30 to 30·5 m.p.h.

Let $\delta v$ be the increment in $v$ and $\delta R$ the resulting increment in $R$.

Then
$$R+\delta R = \frac{(v+\delta v)^2}{10} = \frac{v^2+2v\,\delta v+(\delta v)^2}{10},$$

and
$$\delta R = \frac{2v\,\delta v+(\delta v)^2}{10}.$$

Putting $v = 30$, $\delta v = 0{\cdot}5$
$$\delta R = \frac{30+0{\cdot}25}{10} = 3{\cdot}025.$$

The increase of air resistance is 3·025 lb. wt.

Now the rate of increase of $R$ with $v$ is $2v/10 = \frac{1}{5}v$. This rate varies with $v$ but if $\delta v$ is small it cannot alter much as $v$ changes to $v+\delta v$. Let us suppose that the rate of increase of $R$ with $v$ is held constant at $\frac{1}{5}v$ as $v$ increases from $v$ to $v+\delta v$. On this assumption the change in $R$ is $\frac{1}{5}v\times\delta v$. Putting $v = 30$ and $\delta v = 0{\cdot}5$ we have $\frac{1}{5}v\,\delta v = 3$. The difference between the change of $R$ calculated by this approximate method and its true value is only 0·025 or $\frac{1}{40}$ lb. wt. and this is quite insignificant in

comparison with the forces acting on the car The practical man would therefore be justified in using the simpler approximate calculation

Now consider this method of approximating in terms of the graph of the formula It amounts to assuming that the change of $R$ may be calculated approximately by replacing the curve $PQ$ by the straight line $PT$ having the same gradient as the curve at $P$, i e the tangent at $P$ We then calculate $TN$ instead of $QN$ That is, we calculate $dR$ instead of $\delta R$

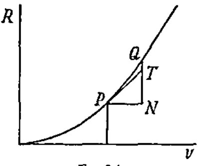



FIG 3 4

We have already seen that

$$\delta R = \frac{2v\,\delta v + (\delta v)^2}{10} = \frac{v\,\delta v}{5} + \frac{(\delta v)^2}{10}.$$

From $R = v^2/10$ we have immediately

$$dR = \frac{2v}{10}dv = \frac{v}{5}dv$$

But $dv$ is $\delta v$ and when $\delta v$ is small, $(\delta v)^2$ is very small (If $\delta v$ is 0 1, $(\delta v)^2 = 0\ 01$ )

Hence $dR$ is the part of $\delta R$ which is linear in $\delta v$ and it may be described, roughly, as the most important part of $\delta R$ when $\delta v$ is small In practical applications of formulae it is often useful to calculate the increment of the dependent variable approximately We postpone the important question of how accurate the approximation is, merely observing that its accuracy is likely to increase as the increment in the independent variable decreases

EXAMPLE Calculate $(10\ 1)^2$ approximately

*Solution* We know the square of 10 and we require the square of 10 1 We therefore write down the formula giving the square ($y$) of a number ($x$),

$$y = x^2$$

We require $y + \delta y$ when $x = 10$ and $\delta x = 0\ 1$

Now
$$\begin{aligned} dy &= 2x\,dx \\ &= 20 \times 0\ 1 \quad \text{when } x = 10 \text{ and } dx = 0\ 1 \\ &= 2 \end{aligned}$$

Therefore $\delta y = 2$

But $y = 100$ Therefore $y + \delta y = 102$

[Find the exact value of $(10\ 1)^2$ and the error of the approximation ]

## Exercise 3.C

1. If $n$ is a number and $c$ is its cube, $c = n^3$.

(i) Find the average rate of increase of $c$ with $n$ from $n = 1$ to $n = 1{\cdot}2$. (ii) Find the rate of increase of $c$ with $n$ when $n = 1$. (iii) If $c$ is increased constantly at the rate (ii), what would be its value when $n = 1{\cdot}2$? What is the error of this approximation to $(1{\cdot}2)^3$?

2. If $y = x^3$, complete the following table:

| $x = 1.$ $\delta x$ | 1 | 0·1 | 0·01 |
|---|---|---|---|
| $\delta y$ | | | |
| $dy$ | | | |
| $\delta y - dy$ | | | |

State approximations to $(1{\cdot}1)^3$ and $(1{\cdot}01)^3$ and their errors.

3. The weight, $W$ lb., of one fathom of 6-strand rope is given approximately by the formula $W = x^2$, where $x$ is the circumference of the rope in inches. What is the weight of one fathom of a rope whose circumference is 4 in.? Find approximately the weight of one fathom of a rope whose circumference is $4\frac{1}{8}$ in.

4. The total road and air resistance on a motor-car of ordinary design weighing 1 ton is given as $R = 50+5{\cdot}83(v/10)^2$, where $v$ is the velocity in m.p.h. and $R$ is the resistance in lb. wt.

Calculate approximately the increase in the resistance when the velocity increases from 40 to 42 m.p.h.

5. The safe load, $T$ tons, of beams of a certain cross-section and length $x$ ft. is given by $T = 810/x$. Find approximately the change in the safe load when the length is increased from 27 ft. to 28 ft. 6 in.

6. Calculate approximately $(10{\cdot}1)^4$.

7. Calculate the cube of 41 to the nearest thousand.

8. Calculate $(29{\cdot}8)^2$ approximately.

9. The intensity of illumination, $I$ foot-candles, of a 100-candle-power lamp at a distance $x$ ft. from the lamp is given by $I = 100/x^2$. Find approximately the change in illumination if the distance from the lamp is increased from 10 ft. to 10 ft. 6 in.

10. The weight, $W$ lb., of 1 ft. length of iron pipe of 2-in. bore is given by $W = 10{\cdot}4(2+t)t$, where $t$ is the thickness of the wall of the pipe. When $t = \frac{1}{8}$, $W = 2{\cdot}8$. Find $W$ approximately when $t = \frac{3}{16}$.

11. In considering the strength of girders, engineers use a quantity $I$ defined in terms of the height, $H$, of the girder by $I = H^3/12$. An engineer's pocket book gives the following values of $I$ and $H$:

$$H\ 10 \quad I\ 83{\cdot}3; \quad H\ 10\tfrac{1}{8} \quad I\ \underline{86{\cdot}5}.$$

Calculate the underlined value approximately from the other three.

12 A formula used with girders of circular cross section is $I = (\pi D^4)/32$, where $D$ is the diameter I found the following values of $I$ and $D$ to be

$$D\ 1\tfrac{7}{8} \quad I\ \underline{1\ 2}, \quad D\ 2 \quad I\ 1\ 6$$

Calculate the underlined value approximately from the other three

## 3.4. Approximate determination of error

In the newspapers of 23 February 1933 it was reported that Sir Malcolm Campbell had achieved a land speed record of 272 108 m p h , this being the average speed on two runs over a measured mile The results of the runs were given as

| | *Speed (m p h )* | *Time (sec )* |
|---|---|---|
| Southbound | 273 556 | 13 16 |
| Northbound | 270 676 | 13 30 |

If a distance of 1 mile is covered in $t$ sec $= t/3600$ hr , the speed, $v$ m p h , is given by

$$v = \frac{3600}{t}.$$

Putting $t = 13\ 16$, we find for the southbound run

$$v = \frac{3600}{13\ 16} = \frac{360000}{1316}$$
$$= 273\ 556$$

```
         273 556
1316)360000 000
     2632
      9680
      9212
       4680
       3948
        7320
        6580
         7400
         6580
          8200
          7896
           304
```

By continuing the division it is possible to obtain still more figures If we do this, do we find the speed more accurately? Not necessarily, because the calculation does not start from exact values of the distance and time All measurements are

approximate and there would undoubtedly be errors in the measurement of the mile and the time of the run. Suppose we neglect the error in the measurement of the distance and assume that the time was measured correct to the nearest $\frac{1}{100}$ sec. so that the greatest error in $t$ is $\frac{1}{200}$ sec. What effect does this error have on the calculated value of $v$? This is easily found approximately by the method of section 3.3.

We have $$v = \frac{3600}{t}.$$

Therefore $$dv = -\frac{3600}{t^2}dt$$

and $$\delta v \doteqdot -\frac{3600}{t^2}.\delta t.$$

Substituting $t = 13{\cdot}16$ and $\delta t = \pm\frac{1}{200}$ (we do not know whether the true time is greater or smaller than the observed time), we have
$$\delta v \doteqdot \pm\frac{3600}{(13{\cdot}16)^2}\times\frac{1}{200}.$$

We approximate still further, replacing $13{\cdot}16$ by 13.
$$\delta v \doteqdot \pm\frac{18}{169} \doteqdot \pm 0{\cdot}1.$$

Therefore the true value of $v$ may be any number between approximately $273{\cdot}556+0{\cdot}1$ and $273{\cdot}556-0{\cdot}1$, i.e. between $273{\cdot}656$ and $273{\cdot}456$.

It is now clear that the last two figures in the quotient of our division have no meaning at all. It is better to stop at the first decimal figure and give the answer as 273·6 m.p.h. within about 0·1 m.p.h.

Thus the figures given in the newspaper were misleading since they imply a greater accuracy in the determination of the speed than is justified by the accuracy with which the time was measured.

Whenever the result of an experiment is calculated from certain observed data by means of a formula, it is important to estimate, in advance, the error of the result for two reasons:

(1) to save the labour of working out meaningless figures;
(2) to avoid giving a false impression of the accuracy of the result.

In general this error is caused by errors in a number of measurements, but in the following exercise we limit ourselves to the determination of the error caused by an error in only one measurement.

## Exercise 3 D

1 A farmer 'steps out' the side of a square field and finds it to be 220 yd with a possible error of 10 yd He works out the area of the field in acres from the formula $A = x^2/4840$ and says it is 10 acres Show that he is really only justified in saying that the area is between 9 and 11 acres

2 The weight of 100 yd of steel wire of diameter $t$ in is given approximately by the formula $W = 774t^2$ If the maker guarantees the accuracy of $t$ to $\frac{1}{1000}$ in (possible error $\frac{1}{2000}$ in ), to what accuracy are we justified in calculating $W$ from the formula (a) when $t = 0\ 3$, (b) when $t = 0\ 02$ ?

3 By the side of British railway tracks, white posts are set up at intervals of $\frac{1}{4}$ mile A boy travelling in a train estimates that he takes 15 sec to pass from one post to the next with a possible error of $\frac{1}{2}$ sec Find (i) the formula for the speed $v$ m p h when the observed time is $t$ sec , (ii) the approximate error in the calculated speed

4 Suppose an apparatus were designed to measure the time of a racing car over a measured mile to the nearest $\frac{1}{1000}$ sec Find, to as many figures as are justified, the speed of a car whose time is 14 000 sec

5 A 100 yd race is timed by stop watch to the nearest $\frac{1}{5}$ sec Determine approximately between what limits the velocity (in f p s ) of a runner lies if his time is 10 sec dead

6 A boy sets out to find the value of $g$ (the acceleration of gravity) by observing the time of swing, $T$ sec , of a simple pendulum of length 20 ft The formula he uses is $g = 80\pi^2/T^2$ He finds $T = 5$ sec with a possible error of $\frac{1}{5}$ sec Determine approximately the error in the calculated value of $g$ (Take $\pi^2 = 10$ )

7 The Thames tonnage, $T$, of a yacht is calculated from the formula $T = (L-B)B^2/188$, where $L$ ft is the length and $B$ ft is the greatest breadth of the yacht The designer draws the lines of a yacht so that $L = 20$, $B = 7$, but the builder may be expected to get $L$ correct and $B$ in error by as much as 0 1 ft With $L = 20$, $B = 7$ the formula gives $T = 3\ 39$ Show that the tonnage of the completed yacht may be expected to lie between 3 32 and 3 46

8 By regarding the earth as a sphere of radius 3,957 miles and using the formula $A = 4\pi r^2$, I calculate that the surface area is 196,750,000 sq miles But it is known that the earth is not a true sphere Assuming that the assumption which I made is equivalent to an uncertainty in $r$ of $\pm 4$ miles find the possible error in the calculated surface area and give the answer to as many figures as are justified

### 3.5. A notation for the gradient function

Let $P = (x, y)$ be any point on a curve and let $PT$ be the tangent at $P$. Give $x$ any increment $dx$ which is represented in Fig. 3.5 by $HK = PN$. Then $dy$ is represented by $TN$. Therefore the gradient at $P = TN/PN = dy/dx$.

Since this holds at all points of the curve at which there is a gradient, we use $dy/dx$ as a convenient notation for the gradient function. Thus if the curve is $y = x^2$ the gradient function is

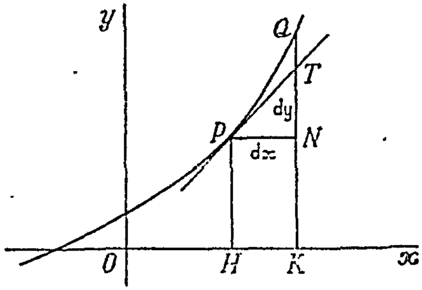



FIG. 3.5

denoted by $dy/dx = 2x$, and the gradient at the point $P \equiv (x_1, y_1)$ is given by

$$\frac{dy}{dx} = 2x_1.$$

EXAMPLE. Find the gradient of the tangent at (1, 1) to the curve

$$y = 3x^2-4x+2.$$

*Solution.* We have $$\frac{dy}{dx} = 6x-4.$$

When $x = 1$ $$\frac{dy}{dx} = 6-4$$
$$= 2.$$

The gradient of the tangent at $x = 1$ is 2.

Similarly if $p$ is given in terms of $v$ by the formula $p = 10/v$, the rate of change of $p$ with $v$ is denoted by $dp/dv$. Thus

$$dp/dv = -10/v^2.$$

In a distance-time formula the rate of change of distance with time is denoted by $ds/dt$. Hence $ds/dt = v$, the velocity at time $t$. Thus if $s = 6t^2-14t$

$$\frac{ds}{dt} = v$$
$$= 12t-14.$$

## 3.6. The normal

The normal at any point, $P$, of a curve is the straight line through $P$ at right angles to the tangent at $P$.

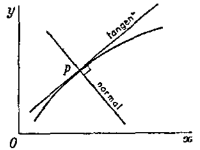



FIG 3 6

EXAMPLE Find the equation of the normal at (2, 8) to $y = x^3$.

*Solution* Since
$$y = x^3$$
$$\frac{dy}{dx} = 3x^2$$

The gradient of the tangent at $x = 2$ is 12 Therefore the gradient of the normal is $-\frac{1}{12}$ and its equation is

$$12y+x = 12\times8+2 \text{ (it passes through the point (2, 8))}$$
$$= 98$$

EXERCISE 3 E

1 If $y = 8x^2-5x+4$ find $dy/dx$ in terms of $x$ What is the gradient of the curve when $x = 3$ ?

2 Find the gradients of the tangent and normal to $y = x^3-4x$ at (2, 0)

3 If $s = 4t^3$, find $ds/dt$ If $s$ is the distance travelled by a body in time $t$, find the velocity when $t = 2 5$

4 If $w = 30h^3+2h^2-5$, find the rate of change of $w$ with $h$ when $h = 1$

5 Find the equation of the normal at (1, 1) to $y = 1/x$ Show that it passes through the origin

6 Find the equation of the normal to $8y = x^2$ at the point $(-6, 4\frac{1}{2})$

7 Find the equations of the tangent and normal at (a) (1, 0), (b) (0, 1) to $y+x^3 = 1$

Find the coordinates of the vertices of the quadrilateral formed by these four lines and the radius of the circle which passes through their vertices

8. Find the equations of the tangent and normal at (2, 4) to the curve $y = x^2$. If the tangent and normal meet the $y$-axis at $P$, $Q$ prove that the middle point of $PQ$ is $(0, \frac{1}{4})$.

9. The normal to $xy = 3$ at $(2, \frac{3}{2})$ meets the axes at $A$ and $B$. Find the length of $AB$.

10. $y = (x-1)^2$. Complete the following table at $x = 2$.

| $x = 2$, $\delta x$ | 0·5 | 0·1 | 0·01 |
|---|---|---|---|
| $\delta y$ | | | |
| $dy$ | | | |
| $\dfrac{\delta y}{\delta x}$ | | | |
| $\dfrac{dy}{dx}$ | | | |

(a) What is the gradient of the curve at $x = 2$?
(b) What is the average gradient from $x = 2$ to $x = 2·01$?

11. If $s = t^3-3t^2+2t-5$, find the formula for $v$ in terms of $t$. How do you represent $v$ by a symbol involving $s$ and $t$? Find $v$ when $t = 1$.

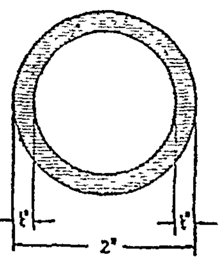
12. A stone is thrown vertically into the air and its height, $h$ ft., above the ground after $t$ sec. is given by $h = 100t-16t^2$. Find its velocity after (a) 2 sec., (b) 4 sec., and state whether it is rising or falling.

13. The area, $A$ sq. in., of the cross-section of a pipe whose overall diameter is 2 in. and whose wall is $t$ in. thick is given by

$$A = \pi(2t-t^2).$$

Find the rate of increase of $A$ with $t$ when

$$t = (a)\ \tfrac{1}{8} \text{ in.},\ (b)\ \tfrac{7}{8} \text{ in.}$$




14. Find the equations of both the tangents to $y = 2x+1/x$ which are parallel to $y = x$.

15. Find the equations of all the normals to $y = 5x/2+1/x$ which have gradient $\frac{2}{3}$.

16. Find the equation of the tangent at the origin to

$$y = 2x-10x^3/3.$$

Find the equations of the tangents to the curve which are perpendicular to the tangent at the origin.

## 3.7. Increase and decrease

If $x$, $y$ are the coordinates of a point on a curve and if $dy/dx$ is positive when $x = x_1$, it is clear that the tangent at $x = x_1$ is a rising line. The rate of increase of $y$ with $x$ is positive when

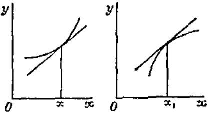



FIG 3 7 $\dfrac{dy}{dx}$ positive when $x = x_1$

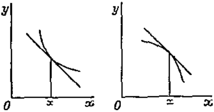



FIG 3 8 $\dfrac{dy}{dx}$ negative when $x = x_1$

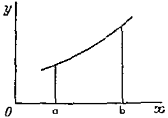



FIG 3 9 $\dfrac{dy}{dx}$ positive from $x = a$ to $x = b$

$x = x_1$ This means that $y$ increases as $x$ increases through the value $x_1$

If $dy/dx$ is negative when $x = x_1$, $y$ decreases as $x$ increases through the value $x_1$

If $dy/dx$ is positive for every value of $x$ from $x = a$ to $x = b$ $(b > a)$ $y$ increases steadily from $x = a$ to $x = b$ and the value of $y$ when $x = b$ must be greater than the value of $y$ when $x = a$ Similarly, if $dy/dx$ is negative for every value of $x$ from $x = a$ to $x = b$, the value of $y$ when $x = b$ is less than the value of $y$ when $x = a$ [Draw a sketch similar to Fig 3 9 for this case ]

*Note* The phrase 'as $x$ increases' is often omitted When nothing is said about the change of $x$ it is assumed to be an increase Thus we say that when $dy/dx$ is positive, $y$ increases and when $dy/dx$ is negative, $y$ decreases

## EXERCISE 3.F

1 If $y = x^2$, does $y$ increase or decrease as $x$ increases through the value 1 ? For what values of $x$ does $y$ (*a*) increase, (*b*) decrease ?

2 If $y = x^3$, show that $y$ does not decrease for any value of $x$

3 $p$ and $v$ are connected by the formula $pv = 100$ Show that when $v$ increases, $p$ decreases $(v \neq 0)$

4 For what values of $x$ does $y = x^3 - 3x + 4$ decrease ?

5 For what values of $x$ does $y = 1/x^2$ increase ?

6–8 Sketch the shape of possible curves satisfying the data of

6 $y = 2$ when $x = 1$, $dy/dx$ negative from $x = 1$ to $x = 4$

7 $y = -1$ when $x = 0$, $dy/dx$ positive from $x = 0$ to $x = 3$

8 $y = 0$ when $x = 0$, $dy/dx$ positive from $x = 0$ to $x = 1$, $dy/dx = 0$ when $x = 1$, $dy/dx$ negative from $x = 1$ to $x = 2$

9. Find the values of $y$ and $dy/dx$ when $x = 0$ and $y = x^4+x-1$. Draw a sketch of the curve $y = x^4+x-1$ for small positive and negative values of $x$.

If the equation $x^4+x-1 = 0$ has a root near 0, is it likely to be positive or negative?

10. If $y = x^2-2x+2$, what is the value of $y$ when $x = 1$? Show that $y$ is greater than 1 for all values of $x > 1$. What is the sign of $dy/dx$ when $x < 1$? Show that $y$ is also greater than 1 when $x < 1$. Sketch the graph.

11. Show that $y = 2x^3-3x^2+1$ is positive when $x > 1$.

12. Show that $2-x-x^2$ is negative when $x > 1$.

## 3.8. Maxima and Minima

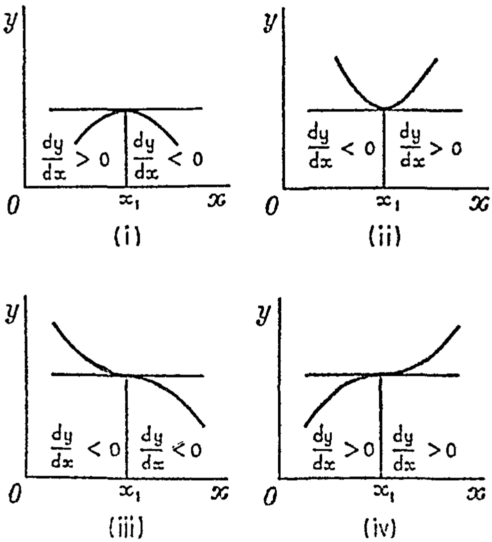



FIG. 3.10

We have seen that $y$ increases or decreases when $x = x_1$ according as $dy/dx$ is positive or negative. If $dy/dx = 0$ at $x = x_1$, it follows that $y$ neither increases nor decreases. For this reason, the values of $y$ when $dy/dx = 0$ are called *stationary values*. The tangent at a point on a curve where $y$ is stationary is parallel to the $x$-axis. Since $dy/dx$ can be positive or negative on either side of $x = x_1$, there are four cases to be considered. These are shown in Fig. 3.10 (i–iv).

In cases (i) and (ii) $dy/dx = 0$ and changes sign at $x = x_1$ but in cases (iii) and (iv) although $dy/dx = 0$ when $x = x_1$, its sign does not actually change.

The points on a curve at which $dy/dx$ changes sign by passing through the value zero are called *turning points* Fig 3 10 (i) and (ii) illustrate the two possible kinds of turning points In case (i) the value of $y$ is said to be a *maximum* and in case (ii) the value of $y$ is said to be a *minimum* Cases (i) and (ii) may be distinguished by the way in which $dy/dx$ changes sign

$y$ is a *maximum* at $x = x_1$, if

(i) $dy/dx = 0$ when $x = x_1$,

(ii) $dy/dx$ changes sign from $+$ to $-$ as $x$ increases through the value $x_1$

$y$ is a *minimum* at $x = x_1$, if

(i) $dy/dx = 0$ when $x = x_1$,

(ii) $dy/dx$ changes sign from $-$ to $+$ as $x$ increases through the value $x_1$

Consideration of cases (iii) and (iv) where the value of $y$ is stationary but not a maximum or minimum, is postponed

EXAMPLE 1 Find the maximum and minimum values of

$$y = 2x^3-3x^2-12x+10$$

*Solution* We have

$$dy/dx = 6x^2-6x-12 = 6(x-2)(x+1)$$

Therefore $dy/dx = 0$ when $x = 2$ or $x = -1$

There are two stationary values of $y$,

(i) when $x = 2$, $\quad y = -10$

(ii) when $x = -1$, $\quad y = 17$

We now have to decide to which of the four cases each stationary value belongs

As $x$ increases through the value 2 $x-2$ changes sign from $-$ to $+$ and $x+1$ is positive all the time since it is nearly 3

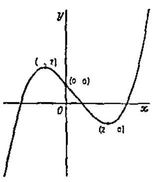



FIG 3 11

Hence $dy/dx = 6(x-2)(x+1)$ changes sign from $-$ to $+$

The shape of the curve near $x = 2$ is 

Therefore $y = -10$ is a minimum value

As $x$ increases through the value $-1$, $x-2$ is near $-3$ and therefore negative, while $x+1$ changes sign from $-$ to $+$

Therefore $dy/dx = 6(x-2)(x+1)$ changes sign from $+$ to $-$

The shape of the curve near $x = -1$ is 

Hence $y = 17$ is a maximum value

We illustrate these results by sketching the graph of

$$y = 2x^3-3x^2-12x+10$$

[Verify that the sign of $dy/dx$ is $+$ when $x < -1$, $-$ when $-1 < x < 2$, $+$ when $x > 2$. Also when $x = 0$, $y = 10$.]

The graph suggests that $y$ has values greater than 17. Putting $x = 10$ we find

$$y = 2000-300-120+10 = 1590.$$

Similarly $-10$ is not the smallest value of $y$. When $x = -10$,

$$y = -2000-300+120+10 = -2170.$$

Thus the particular kind of stationary value which we have called a maximum value (case (i), Fig. 3.10) is not necessarily the greatest value of $y$ but it *is* the greatest value in its own neighbourhood. It might have been better to call it a *local* maximum. Indeed $y$ may have several maximum values. If its graph resembles the section of a range of mountains, the summit of each peak gives a maximum value, regardless of the relative heights of the different peaks. Similarly the bottom of each valley gives a minimum value.

EXAMPLE 2. Find the stationary values of

$$y = \frac{(x-1)(x-4)}{x},$$

distinguishing between maxima and minima.

*Solution.*

$$y = x+\frac{4}{x}-5.$$

$$\frac{dy}{dx} = 1-\frac{4}{x^2}.$$

Stationary values occur when $1-4/x^2 = 0$, i.e. when $x = \pm 2$.

Consider $x = 2$.

[$x$ just less than 2; $4/x^2$ greater than 1; gradient negative. $x$ just greater than 2; $4/x^2$ less than 1; gradient positive.]

At $x = 2$ the gradient changes sign from $-$ to $+$.

The shape of the graph in the neighbourhood is $\smile$.

Therefore $x = 2$ gives a minimum value of $-1$.

Now consider $x = -2$.

[$x$ just less than $-2$ (e.g. $-2{\cdot}1$); $4/x^2$ less than 1; gradient positive. $x$ just greater than $-2$; gradient negative.]

The gradient changes sign from $+$ to $-$.

The shape of the graph is $\frown$.

Therefore $x = -2$ gives a maximum value of $-9$.

Here we have a remarkable result, a maximum value smaller than a minimum value!

To see that this makes sense, we draw the graph.

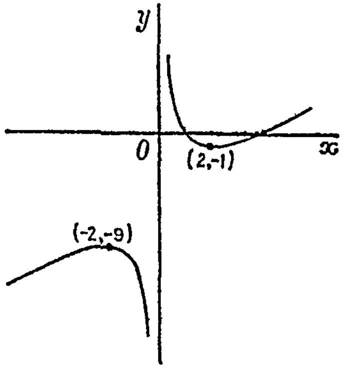



FIG. 3.12

This example emphasizes the necessity of distinguishing between maximum and minimum values by discussing the sign change of $dy/dx$ and not by relying on the relative magnitudes of the stationary values

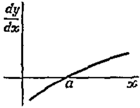
EXERCISE 3 G

1 Find the maximum value of $y = 4x-x^2$ Is it the greatest value of $y$?

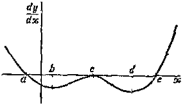
2–6 Find the stationary values of the following functions Distinguish between maxima and minima

2 $x^2+6x+8$ 3 $1-2x-x^2$ 4 $x^2+10$

5 $(1-x)^2$ 6 $3x-x^3-\frac{1}{4}$




Ex 3 G 7




Ex 3 G 8

7 The figure is a sketch of the graph of $dy/dx$ in the neighbourhood of $x = a$ Is $y$ a maximum or minimum when $x = a$? Sketch a possible graph of $y$ in the neighbourhood of $x = a$, given that $y = 0$ when $x = a$

8 The figure is a sketch of the graph of $dy/dx$ Which of the values of $x$ named on the sketch give stationary values of $y$? Which gives (i) a maximum, (ii) a minimum value of $y$? Sketch a possible graph of $y$, given that it passes through the origin and mark $a, b, c, d, e$ on your sketch

9–16 Find the maximum and minimum values of the functions

9 $x+1/x$ 10 $x^2/2+1/x$ 11 $x^3-3x^2$

12 $x^3-3x+1$ 13 $3+3x^2-x^3$ 14 $4x(x^2-3)$

15 $\frac{3}{2}x^2+24/x$ 16 $\frac{1}{3}(x^3+40)+16/x$ 17 $x^2+1/x^2$

## 3 9. The gradient of the gradient function

Suppose we wish to draw the graph of

$$y = 2x^3-3x^2$$

The gradient function is

$$\frac{dy}{dx} = 6x^2-6x$$

and we have found in section 2 13 that the graph of the gradient is useful in drawing the graph of $y$ In the present case the

gradient function is of the second degree and in drawing its graph the gradient of the gradient function will clearly be useful. If we denote $dy/dx$ by $g$, we have

$$g = 6x^2-6x,$$

$$\frac{dg}{dx} = 12x-6.$$

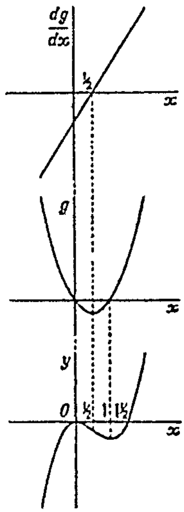


FIG. 3.13

Hence the graph of $dg/dx$ is a straight line and it may be drawn easily. We find $dg/dx = 0$ when $x = \frac{1}{2}$, and from the graph it is seen that $dg/dx$ changes sign from $-$ to $+$. Hence $x = \frac{1}{2}$ is a minimum point on the graph of $g = 6x^2-6x$ which has no other stationary values. When $x = \frac{1}{2}$, $g = -1\frac{1}{2}$. The graph of $g = 6x^2-6x$ may now be drawn.

This graph crosses the $x$-axis at $x = 0$ and $x = 1$. At $x = 0$, $g = dy/dx$ changes sign from $+$ to $-$. Hence $y$ has a maximum value when $x = 0$. This maximum value is 0.

At $x = 1$, $dy/dx$ changes sign from $-$ to $+$. Hence $x = 1$ gives a minimum value of $y$. This minimum value is $-1$.

There are no other stationary values of $y$. The graph of $y$ rises when $x$ is negative and when $x$ is greater than 1. Between $x = 0$ and $x = 1$, the graph falls and its steepest negative gradient is at $x = \frac{1}{2}$.

## EXERCISE 3.H

1. Find the gradient and the gradient of the gradient at any point of $y = x^3-3x$.

Sketch the graph of this equation.

2. Sketch the graph of $y = x^3+x$.
3. Sketch the graph of $y = 4x-\frac{1}{3}x^3$.
4. Sketch the graph of $y = x^3+3x^2$.

5–7. Find the gradient of the gradient function:

5. $x^2$. 6. $x-1/x$. 7. $\pi(1+2x-x^4)$.

8–14. Denote the gradient function by $g$ and find $g$ and $dg/dx$:

8. $10x^4-8x^2$. 9. $6x^2-4/x$.
10. $2x+17$. 11. $x^4-2x+5$.
12. $x(x-1)(x-2)$. 13. $(1-x)^2/4x$.
14. $(1+x)^2/x^2$.

## 3 10. A notation for the gradient of the gradient function

In section 3 9 we used the notation

$$y = 2x^3-3x^2,$$

$$\frac{dy}{dx} = g = 6x^2-6x,$$

$$\frac{dg}{dx} = 12x-6$$

Since $g$ is $dy/dx$ we may write this

$$\frac{d(dy/dx)}{dx} = 12x-6$$

For ease of writing, this is condensed to

$$\frac{d^2y}{dx^2} = 12x-6$$

*The gradient of the gradient of $y$ is denoted by $d^2y/dx^2$*

There is, of course, a reason for writing this symbol as a fraction in just this form But it is not convenient or necessary to explain here what meanings the numerator and denominator have if they are taken separately For the present we always use the symbol as a whole to indicate concisely that the operation of taking the gradient has been performed first upon $y$ and then upon its gradient

### Exercise 3 J

1–10 Find $d^2y/dx^2$

1 $y = 10x^4+12x^3$ 2 $y = x^2(x^4/15-1)$

3 $10y = x^{10}+x^5$ 4 $y = (x^3+1)/x^4$

5 $dy/dx = 8x-1/x$ 6 $y = x(x^2+1/x^2)$

7 $y = x^4+4x^2+x$ 8 $y = ax^5$ [$a$ is a constant]

9 $dy/dx = \frac{1}{10}(x^5-5/x)$

10 $y = ax^2+bx+c$ [$a$, $b$ $c$ are constants]

11 (*a*) If $y$ and $x$ are connected by a formula, write down the mathematical notation for (i) the rate of change of $y$ with $x$, (ii) the rate of change of the rate of change of $y$ with $x$

(*b*) If $W$ and $l$ are connected by a formula write down the mathematical notation for (i) the rate of change of $W$ with $l$, (ii) the rate of change of the rate of change of $W$ with $l$

12 The weight, $W$ lb, of a fathom of six strand rope of circumference $c$ in is given by $W = c^2$ Show that the rate of change of the rate of change of $W$ with $c$ is constant and find its value

13. The weight of coal in a locomotive's tender after it has travelled $x$ miles is $(15000-42x)$ lb. Show that the rate of change of the rate of consumption of coal is zero. Find the consumption of coal in lb. per mile.

14. A formula for the load, $P$, supported by a column of height $l$ is $P = k/l^2$, where $k$ is a constant. Find

(a) the rate of change of $P$ with $l$,

(b) the rate of change of the rate of change of $P$ with $l$.

15. On board the S.S. —— was a second mate who had studied mathematics in his youth. At the inquiry into the loss of the ship the following extract from his diary was read. It was explained that the symbol $h$ was used to denote the depth of water in the well of the ship's hold.

| | | |
|---|---|---|
| 9 a.m. | Sounded the well. $dh/dt, +$. Started the pumps. | |
| 10 a.m. | $dh/dt, -. d^2h/dt^2, 0$. | |
| 11 a.m. | $dh/dt, +. d^2h/dt^2, +$. | Distress signals sent out. |
| 3 p.m. | abandoned ship. | |

Write this extract from the diary in plain English.

16. Find $dy/dx$ when $d^2y/dx^2 = x$, and $dy/dx = 1$ when $x = 1$.

17. If $d^2y/dx^2 = 2+4x$ and $dy/dx = 10$ when $x = 2$, find $dy/dx$.

18. Find $y$ if $d^2y/dx^2 = 6x^2$ and both $dy/dx$ and $y = 1$ when $x = 1$.

19. Find $y$ when $d^2y/dx^2 = 1/x^3$ and $y = 0$, $dy/dx = 2$ when $x = -1$.

20. Sketch the curve for which $d^2y/dx^2 = \frac{1}{2}$ if it passes through $(0, 2)$ and has gradient 1 at that point.

## 3.11. The use of $d^2y/dx^2$ to distinguish maxima and minima

Suppose that at $x = x_1$, $dy/dx = 0$ and $d^2y/dx^2$ is positive. Then $dy/dx$ is increasing at $x = x_1$. ($d^2y/dx^2$ is the gradient of $dy/dx$ and is positive.) Therefore $dy/dx$ changes sign at $x = x_1$ from $-$ to $+$, and in the neighbourhood of $x = x_1$ the graph of $y$ has the shape shown in Fig. 3.14.

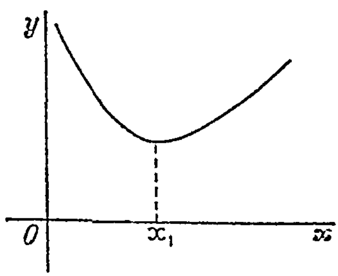


FIG. 3.14

Hence $y$ has a minimum value at $x = x_1$.

Similarly if $dy/dx = 0$ and $d^2y/dx^2$ is negative at $x = x_1$, we can show that $y$ has a maximum value. (A separate figure should be drawn for this case.)

We now have the following test for deciding whether a stationary value is a maximum or minimum

*If* $dy/dx = 0$ *and* $d^2y/dx^2$ *is* $\begin{Bmatrix}\textit{positive}\\ \textit{negative}\end{Bmatrix}$ *when* $x = x_1$ *then* $y$ *is a* $\begin{Bmatrix}\textit{minimum}\\ \textit{maximum}\end{Bmatrix}$ *for this value of* $x$

This test will usually be found easier to apply than the test by discussion of the sign change of $dy/dx$ given in section 3 8 However, it cannot be relied upon entirely because it may happen that $d^2y/dx^2 = 0$ at $x = x_1$ In that case, not only does the test fail to distinguish a maximum from a minimum value but it does not even show that the stationary value is a maximum or minimum For, although $dy/dx = 0$ at $x = x_1$, we cannot now prove that it changes sign and the graph may have one of the shapes illustrated in Fig 3 10 (iii, iv)

When $dy/dx = d^2y/dx^2 = 0$ the nature of the stationary value must be decided by considering the sign change of $dy/dx$ This test never fails

EXAMPLE 1 Find the maximum and minimum values of

$$y = 2x^3-3x^2-12x+10$$

(This is Example 1 of section 3 8 and is worked again as a comparison of the two tests )

*Solution* We have

$$\frac{dy}{dx} = 6x^2-6x-12$$
$$= 6(x-2)(x+1),$$
$$\frac{d^2y}{dx^2} = 12x-6$$

$dy/dx = 0$ when $x = 2$ or $x = -1$

These are the stationary values To distinguish them we have,

when $x = 2$, $\qquad \dfrac{d^2y}{dx^2} = 12 \cdot 2-6$
$$= 18$$

[$dy/dx$ increases, i e changes sign $-$ to $+$, shape of graph $\smile$, minimum ]

When $x = -1$, $\qquad \dfrac{d^2y}{dx^2} = -12-6$
$$= -18$$

[$dy/dx$ decreases, i.e changes sign $+$ to $-$, shape of graph $\frown$, maximum ]

Hence $x = 2$ gives the minimum value of $y = -10$ and $x = -1$ gives the maximum value of $y = 17$

EXAMPLE 2. Find whether $y = x^4$ is a maximum or minimum when $x = 0$.

*Solution.*
$$\frac{dy}{dx} = 4x^3,$$
$$\frac{d^2y}{dx^2} = 12x^2.$$

When $x = 0$, $dy/dx = 0$ and $d^2y/dx^2 = 0$. Hence we must consider the sign change of $dy/dx$ when $x = 0$. When $x$ is negative, $4x^3$ is negative and when $x$ is positive, $4x^3$ is positive. Hence $dy/dx$ changes sign from $-$ to $+$.

The shape of the curve near $x = 0$ is .

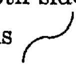
Hence $y = x^4$ has a minimum value at $x = 0$.

EXAMPLE 3. Examine the stationary values of $10x^3-3x^{10}$.

*Solution.* Let
$$y = 10x^3-3x^{10}.$$
Then
$$\frac{dy}{dx} = 30x^2-30x^9.$$
$dy/dx = 0$ when $x^2-x^9 = 0$, i.e. when $x = 0$ or 1.
$$\frac{d^2y}{dx^2} = 60x-270x^8.$$
When
$$x = 0,$$
$$\frac{d^2y}{dx^2} = 0.$$
When
$$x = 1,$$
$$\frac{d^2y}{dx^2} = 60-270$$
$$= -210.$$

This is inconclusive when $x = 0$; when $x = 1$, $dy/dx$ is decreasing, the shape of the curve is , and $y$ is a maximum. To discuss $x = 0$, consider $dy/dx = 30x^2-30x^9$.

In the neighbourhood of $x = 0$, $x^9$ is smaller than $x^2$ so that near the origin, $dy/dx$ has the sign of $30x^2$. Therefore $dy/dx = 0$ at the origin but is positive on both sides of the origin.

The shape of the curve is .

Hence the origin is neither a maximum nor a minimum.

This result may be confirmed by considering an approximation to the equation of the curve near the origin.

When $x$ is small, $y = 10x^3-3x^{10} \doteqdot 10x^3$. [If $x = \frac{1}{10}$, $3x^{10} = 3.10^{-10}$.] Therefore, in the neighbourhood of the origin, the graph of $y = 10x^3-3x^{10}$ is indistinguishable from the graph of $y = 10x^3$, which, as we know, does not have a maximum or minimum value at the origin.

EXERCISE 3 K

1–5 Determine whether the stated value of $x$ gives a maximum or minimum (or neither) and find the maximum or minimum value of $y$ when it exists

1 $x = 2 \quad y = x^4+4x^3-4x^2-64x+8$

2 $x = -1, \quad y = \dfrac{x^9+7}{9} - \dfrac{1}{6x^6}$ 3 $x = 0 \quad y = x^5$

4 $x = 0, \quad y+x^6 = 0$ 5 $x = 1, \quad y = x^{10}+1/x^{10}$

6–18 Determine the maximum and minimum values of $y$

6 $y = 2x^3-15x^2+24x+13$ 7 $y = x^4-4x+5$

8 $y = 5+24x+6x^2-4x^3$ 9 $y = 9(x^3+x^2-x-1)$

10 $y = (4-x)(x^2-3)$ 11 $y = x^2/4+4/x$

12 $y = 5x^2-16x^5$ 13 $y = x-125x^5$ 14 $4y = x^2+16/x^2$

15 $y = 3x(x-2)^2-3$ 16 $2y = 8x^3-12x^2-18x-5$

17 $x^2y = 27x^3+4$ 18 $y = x^3+x^{10}$

## 3 12 Problems involving maxima or minima

We are now able to solve a number of problems in which we are required to find the maximum or minimum value of a quantity depending on one variable

EXAMPLE 1 Find the least value of the sum of a positive number and its reciprocal

*Solution* Let $x$ denote any positive number and let the sum of this number and its reciprocal be $y$ Then $y = x+1/x$

Hence
$$\frac{dy}{dx} = 1-\frac{1}{x^2}, \tag{1}$$
$$\frac{d^2y}{dx^2} = \frac{2}{x^3} \tag{2}$$

From (1) we see that stationary values of $y$ occur when $x = \pm 1$ As we are only interested in positive values of $x$, we substitute $x = 1$ in (2) and find that when $x = 1$, $d^2y/dx^2$ is positive $\left[\underset{x=1}{\smile}\right]$

Hence $x = 1$ gives a minimum value $y = 2$ Now in section 3 8 we noted that the minimum value is not necessarily the least value of $y$ However, by drawing a sketch of the graph of $y = x+1/x$ we see that in this example, the least value of $y$ for positive values of $x$ is its minimum value

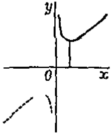



FIG 3 15

Hence the least value of the sum of a positive number and its reciprocal is 2

EXAMPLE 2(*a*). A rectangular box with square base and no lid is to be constructed so that its surface area is 75 sq. in. and its volume is a maximum. Find the dimensions of the box.

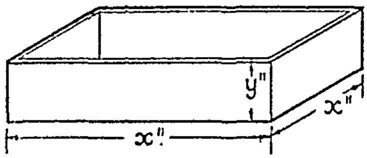


FIG. 3.16

*Solution.* Let the length of each side of the base be $x$ in. and the height $y$ in. Let the volume be $V$ cu. in.

Then
$$V = x^2 y \tag{1}$$
and
$$75 = x^2+4xy. \tag{2}$$

We have to find the maximum value of $V$, and here we meet a difficulty; $V$ is a function of two variables $x$ and $y$ so that we cannot find the gradient function. We must therefore use (2) to eliminate $x$ or $y$ from (1).

Inspecting (2) we see that it is linear in $y$ and that
$$y = \frac{75-x^2}{4x}.$$
Substituting in (1) we have
$$V = \frac{x^2(75-x^2)}{4x} = \frac{75x-x^3}{4} \quad (x \neq 0).$$
$V$ is now expressed as a function of a single variable and we can proceed in the ordinary way:
$$\frac{dV}{dx} = \frac{75-3x^2}{4},$$
$$\frac{d^2V}{dx^2} = -\tfrac{3}{2}x.$$
For a stationary value
$$75-3x^2 = 0,$$
$$x^2 = 25,$$
$$x = 5 \text{ or } -5.$$
$x = -5$ is clearly inapplicable and when $x = 5$
$$\frac{d^2V}{dx^2} = -\frac{15}{2} \quad \left[\begin{smallmatrix} x=5 \\ \frown \end{smallmatrix}\right].$$
Therefore $x = 5$ gives a maximum value of $V$.

But, when
$$x = 5$$
$$y = \frac{75-x^2}{4x}$$
$$= \frac{50}{20} = 2\tfrac{1}{2}.$$
The side of the base must be 5 in. and the height $2\frac{1}{2}$ in. for the volume to be a maximum at $62\frac{1}{2}$ cu. in.

The problem can be set in a more general form.

EXAMPLE 2(b) A rectangular box with square base and no lid is to be constructed so that its surface area is $A$ and its volume is a maximum Prove that the height must be half the side of the square base

*Solution* We now have $V = x^2y$

and $$A = x^2+4xy$$

from which $$y = \frac{A-x^2}{4x},$$

so that $$V = \frac{x^2(A-x^2)}{4x} = \frac{Ax-x^3}{4} \quad (x \neq 0)$$

where $A$ is a constant

Hence $$\frac{dV}{dx} = \frac{A-3x^2}{4},$$

and $$\frac{d^2V}{dx^2} = -\frac{3x}{2}$$

Hence $V$ has a stationary value when $\frac{1}{4}(A-3x^2) = 0$, i e when $A = 3x^2$ If the positive root of this equation is taken, $d^2V/dx^2$ is negative Therefore the positive square root of $\frac{1}{3}A$ gives a maximum value of $V$

We could now find $x$ and $y$ in terms of $A$, but it is more convenient to proceed as follows

We have $x^2 = \frac{1}{3}A$ and $A = x^2+4xy$ and we require a relation between $x$ and $y$ We must therefore eliminate $A$

$$A = 3x^2 \quad \text{and} \quad A = x^2+4xy$$

Therefore $$2x^2 = 4xy$$
$$x = 2y, \quad \text{since } x \neq 0$$

Therefore the side of the base must be twice the height of the box for maximum volume

## EXERCISE 3 L

1 Find the least value of the sum of a positive number and a quarter of its reciprocal

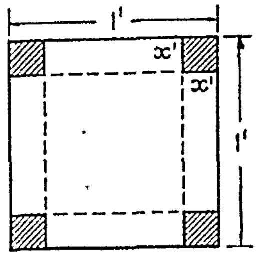
2 $x$ and $y$ are two numbers whose sum is 80 Find $x$ and $y$ when (i) $x^2+y^2$ is a minimum, (ii) $xy$ is a maximum, (iii) $x+y^2$ is a minimum

3 A closed cardboard box has dimensions $x$ in , $2x$ in , $y$ in Find the maximum volume of such a box if the total area of the cardboard is 48 sq in

4 A cylindrical tin with no lid is to have a surface area of $300\pi$ sq in Find the dimensions of the tin when the volume is a maximum

5 A bell tent is to be made in the form of a cone with slant height 12 ft Find the height of the top of the tent when the volume is a maximum

6. The perimeter of a rectangle is $4a$. Prove that the area is a maximum when the rectangle is a square.




7. A sheet of cardboard 1 ft. square has the small shaded squares shown in the figure cut away. The remainder is then folded along the dotted lines to form a box without a lid. Find a formula for the volume of the box in terms of $x$. Find $x$ when the volume is a maximum.

8. For a parcel to be sent through the Post Office its combined length and girth must not exceed 6 ft. If the end section is a square of side $x$ ft. and the length is $y$ ft., find $x$ and $y$ for the volume to be a maximum (the girth is $4x$ ft.).

9. Find the dimensions of the parcel of maximum volume, which can be sent through the post, if the end section is a rectangle with one side twice the length of the other.

10. Think of two numbers whose difference is 1. Note down the value of the sum of the larger and the reciprocal of the smaller. Show that in all cases the answer must either be greater than 3 or less than $-1$. [Denote the smaller number by $x$.] What can you say about the answer if the numbers are restricted to be positive ?

11. (i) A farmer wishes to pasture his sheep by enclosing three sides of a rectangle with hurdles and making use of a straight hedge as the fourth boundary. If he has available 200 yd. of hurdles, what is the greatest area he can enclose ?

(ii) If three sides of a rectangle are composed of hurdles of given total length in the manner of (i), show that the area of the rectangle is greatest when its length is twice its breadth.

12. The dimensions of a closed box are $x$, $2x$, and $y$. It is to be made from a given area of cardboard so that its volume is as great as possible. Find $y$ in terms of $x$.

13. A motorist estimates the cost of running his car as $\frac{1}{600}\left\{14+\frac{v^2}{56}\right\}$ pence per hr., where $v$ is his speed in m.p.h. What is the cost in pence of a 600-mile journey ? Find the most economical speed at which the journey can be performed.

What would be the most economical speed for a journey of $d$ miles ?

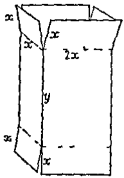
14. A cylindrical tin has radius $r$ in. and height $h$ in. The surface area is to be $A$ sq. in. Show that for the volume to be a maximum

(1) if the tin has no lid, $r = h$;

(2) if the tin is entirely enclosed, $2r = h$.

15 A lidless rectangular box has a volume of $\frac{3}{8}$ cu ft and a base with one side three times as long as the other Find the dimensions of the box when the area of the cardboard of which it is made is least

If the box has a lid and a fixed volume $v$, show that its surface area is least when its height is one and a half times the shorter side of the base




16 A rectangular cardboard box to hold breakfast cereal is to be made so that for a given area of cardboard its volume is a maximum The four sides of the box have flaps at each end These are folded along the dotted lines and gummed to form the top and bottom of the box With the dimensions shown in the sketch show that the box should be made so that $y = 4x$

## 3 13 Further uses of $d^2y/dx^2$

EXAMPLE 1 Sketch possible curves satisfying the following data

(a) When $x = 1$, $y = 1$, $dy/dx = 1$ and $d^2y/dx^2 = 1$

(b) When $x = 1$ $y = 1$, $dy/dx = 1$ and $d^2y/dx^2 = -1$

*Solution* (a) Since $d^2y/dx^2$ is the gradient of $dy/dx$ and is positive, $dy/dx$ increases in the neighbourhood of $x = 1$ When $x$ is a little greater than 1, $dy/dx > 1$, and when $x$ is a little smaller than 1, $dy/dx < 1$

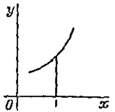



FIG 3 17

The shape of the curve in the neighbourhood of $x = 1$ is therefore that shown in Fig 3 17

(b) Since $d^2y/dx^2$ is negative $dy/dx$ decreases as $x$ increases through the value 1 Hence the shape of the curve in the neighbourhood of $x = 1$ is that shown in Fig 3 18

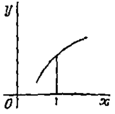



FIG 3 18

If a moving point describes the curves (a) and (b) of Example 1 so that its $x$ coordinate increases through the value 1 in each case the point turns to the left on curve (a) and to the right on curve (b) We say that these curves bend in opposite senses

The sense in which a curve bends is determined by whether the gradient increases or decreases as $x$ increases Consider, for

example, the curve $y = x^2$ which bends in the same sense at all points.

As $x$ increases from 1 to 2, $dy/dx$ increases from 2 to 4. As $x$ increases from $-2$ to $-1$, $dy/dx$ increases from $-4$ to $-2$. In fact, as $x$ increases from a large negative to a large positive value, $dy/dx$ increases steadily from a large negative value, through 0 at $x = 0$, to a large positive value. This is most readily seen by noting that its rate of change, $d^2y/dx^2$, is 2 and is therefore positive at all points.

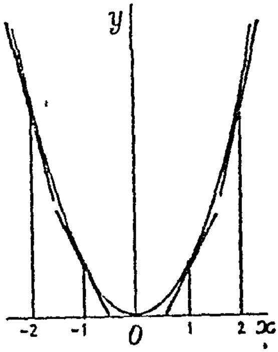



FIG. 3.19

As an example of a curve which bends in the opposite sense we may take $y = -x^2$. For this curve we have $d^2y/dx^2 = -2$. Hence its gradient decreases steadily as $x$ increases.

We now consider the shape of a curve in the neighbourhood of a point $P$ in the four cases which arise according to the signs of $dy/dx$ and $d^2y/dx^2$.

| Sign of $dy/dx$ | Sign of $d^2y/dx^2$ | Change of $dy/dx$ as $x$ increases | Shape of curve near $P$ |
|---|---|---|---|
| (1) + | + | increases | |
| (2) − | + | increases | |
| (3) + | − | decreases | |
| (4) − | − | decreases | |

It is now clear that a curve bends in one sense at all points where $d^2y/dx^2$ is positive, and in the opposite sense at all points where $d^2y/dx^2$ is negative. It follows that at a point $Q$ on a curve where $d^2y/dx^2 = 0$ and changes sign, there is a change in the sense in which the curve bends. If the gradient at $Q$ is positive, the graph therefore takes one of the forms as shown in Fig. 3.20.

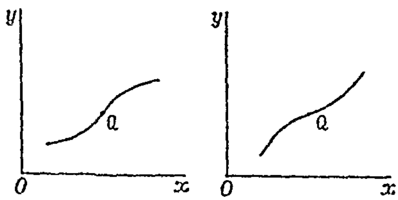



FIG. 3.20

[Draw the two corresponding figures when the gradient at $Q$ is negative.]

The point $Q$ is called a *point of inflexion* on the curve. The gradient at $Q$ may be positive, negative, or zero. Since the

gradient of $dy/dx$ vanishes and changes sign at $Q$ $dy/dx$ is a maximum or minimum at a point of inflexion

We can now complete the investigation of stationary values made in section 3 8 It is clear that cases (iii) and (iv) of Fig 3 10 are points of inflexion with zero gradient

EXAMPLE 2 Show that the origin is a point of inflexion on $y = x^3$

*Solution* We have $dy/dx = 3x^2$ and $d^2y/dx^2 = 6x$ Hence $d^2y/dx^2 = 0$ and changes sign when $x = 0$ Also when $x = 0, y = 0$ Hence the origin is a point of inflexion on the curve

The condition for a point of inflexion at a point $P$ of a curve has been stated in the form $d^2y/dx^2 = 0$ and changes sign at $P$ The necessity for the change of sign may be seen by considering the origin on $y = x^4$

As we have already seen in section 3 11, Example 2, the origin is a minimum point on the curve But, if we differentiate, we find that $d^2y/dx^2 = 12x^2$ and therefore $d^2y/dx^2 = 0$ when $x = 0$ Nevertheless the origin is not a point of inflexion on the curve because $d^2y/dx^2$ does not change sign as $x$ passes through the value 0 and therefore the sense of bending does not alter

EXAMPLE 3 Find the coordinates of the point of inflexion on $y = x^3-3x^2+6x-2$ and sketch the graph

$$\begin{aligned}\frac{dy}{dx} &= 3x^2-6x+6\\ &= 3(x^2-2x+2),\\ \frac{d^2y}{dx^2} &= 6x-6\\ &= 6(x-1)\end{aligned}$$

Therefore $d^2y/dx^2 = 0$ and changes sign at $x = 1$

The point of inflexion is (1, 2) The gradient at this point is given by $dy/dx = 3(1-2+2) = 3$ When $x > 1$, $d^2y/dx^2 > 0$, when $x < 1$, $d^2y/dx^2 < 0$ These determine the way in which the curve bends on either side of (1, 2) The sense of bending cannot change at any other point

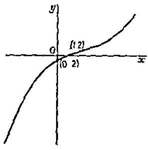


FIG 3 21

When $x = 0, y = -2$

We can now draw the sketch This sketch shows no turning points This may be verified

$$dy/dx = 3(x^2-2x+2) = 3\{(x-1)^2+1\}$$

Hence $dy/dx$ is not 0 for any value of $x$, and there are no stationary values

Also the least value of $dy/dx$ is 3, when $x = 1$

## Exercise 3.M

1. Find the point of inflexion and the turning-points on $y = x^3+3x^2-9x$.

2. Is the origin a point of inflexion on (1) $y+x^5 = 0$, (2) $y+x^6 = 0$? Sketch the curves in the neighbourhood of the origin.

3. Find the point of inflexion on $y = 6x^2-x^3$ and sketch the curve.

4. Find the point of inflexion and turning-points on $y = x^3-3x+1$ and sketch the curve.

5. Show that there are no turning-points or points of inflexion on $y = (1+x^2)/x^2$ and sketch the curve.

6. Find the turning-points and point of inflexion on $3y = 27x-x^3$.

7. Find the points of inflexion on $y = x^3-3/x$ and show that there are no turning-points. Sketch the curve. (You may find it useful to sketch $y = x^3$ and $y = -3/x$ first.)

8. Find the point of inflexion and the turning-point on $x^2y = 1-x$ and sketch the curve.

9. A curve passes through the origin. Sketch a possible form of the curve from $x = 0$ to $x = 1$ if

(a) $dy/dx$ is negative, $d^2y/dx^2$ is positive from $x = 0$ to $x = 1$.

(b) $dy/dx$ is positive, $d^2y/dx^2$ is negative from $x = 0$ to $x = 1$.

10. A curve passes through the origin and from $x = 0$ to $x = 1$ $dy/dx$ is positive and $d^2y/dx^2$ is negative; at $x = 1$, $dy/dx = 1$, $d^2y/dx^2 = 0$; between $x = 1$ and $x = 2$, $dy/dx$ and $d^2y/dx^2$ are both positive. Sketch a curve satisfying these conditions from $x = 0$ to $x = 2$.

11. At (1, 2) on a curve $dy/dx = -\frac{1}{2}$, $d^2y/dx^2 = -1$. Sketch the curve in the neighbourhood of (1, 2).

## 3.14. The motion of a body in a straight line (continued)

Let the distance-time formula for the motion of a body in a straight line be $s = 10+16t^2-2t^4$.

If $v$ denotes the velocity at time $t$

$$v = 32t-8t^3.$$

Now $v$ changes with $t$ and its rate of change is so useful that it has been given a special name.

The rate of change of velocity of a body with the time is called its *acceleration*.

If the acceleration of the body is denoted by $a$, we have

$$a = 32-24t^2.$$

$$\begin{aligned} \text{When } t &= 0, \ a = 32; \\ \text{when } t &= 1, \ a = 8; \\ \text{when } t &= 2, \ a = -64. \end{aligned}$$

A positive acceleration means that $v$ increases with $t$ A negative acceleration means that $v$ decreases and is sometimes called a *retardation*

If $s$ is in ft and $t$ in sec $v$ is measured in f p s The unit of the rate of change of $v$ is therefore f p s per sec This is the unit of acceleration on this system and is often written f p s $^2$

If the velocity of a body changes from $v_1$ at time $t_1$ to $v_2$ at time $t_2$, $(v_2-v_1)/(t_2-t_1)$ is called the *average acceleration* of the body in the interval of time from $t_1$ to $t_2$ Another notation for the average acceleration is $\delta v/\delta t$

Since $a = dv/dt$ and $v = ds/dt$ another notation for the acceleration is $a = d^2s/dt^2$

Example 1 Discuss the motion of a body travelling in a straight line if its distance time formula is $s = t^3-3t^2$ and it starts at $t = 0$

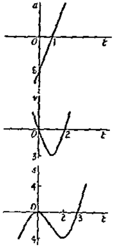


Fig 3 22

*Solution* Since $s = t^3-3t^2$

$$\frac{ds}{dt} = v = 3t^2-6t,$$

$$\frac{d^2s}{dt^2} = a = 6t-6$$

The graphs of $s, v, a$ may now be sketched

It is best to begin with the graph of the acceleration which is a straight line of gradient 6 cutting the $t$ axis where $t = 1$ This is the top graph in Fig 3 22 and it is the graph of the gradient of $v$ By observing the sign change of $a$ at $t = 1$ we see that $v$ has a minimum value when $t = 1$, and from the formula for $v$ this value is $-3$ Also $v = 0$ when $3t(t-2) = 0$, i e when $t = 0$ or 2 The graph of $v$ may now be sketched

But this is the graph of the gradient of $s$ Hence $s$ has stationary values at $t = 0$ and $t = 2$ When $t = 0$, the sign change of $v$ shows that $s$ has a maximum value and this value is 0 Similarly when $t = 2$, $s$ has a minimum value and this value is $-4$ Also $s = 0$ when $t^2(t-3) = 0$, i e when $t = 0$ or $3$ The graph of $s$ may now be sketched

The times at which one at least of $s$, $v$, $a$ vanish are 0, 1, 2, 3 Constructing a table of values at these times, we have

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $s$ | 0 | $-2$ | $-4$ | 0 |
| $v$ | 0 | $-3$ | 0 | 9 |
| $a$ | $-6$ | 0 | 6 | 12 |

The motion may be described as follows At $t = 0$ the body starts from rest with acceleration $-6$ It therefore moves so that $s$ becomes

negative and at $t = 1$, $s = -2$ and $v = -3$, but $a = 0$. At this instant the velocity is a minimum (the speed is a maximum). At $t = 2$, $s = -4$, and $v = 0$. However, $a = 6$, so the body is only at rest for an instant and is, in fact, reversing its direction of motion.

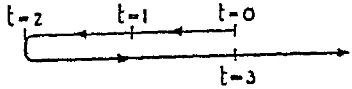



FIG. 3.23

When $t = 3$, it is back at its starting-point, which it passes with velocity 9 and acceleration 12. $s$, $v$, $a$ do not vanish any more and so the body moves away with increasing velocity and acceleration in the direction in which $s$ is measured positively.

## EXERCISE 3.N

*Nos. 1–9 refer to the motion of a body in a straight line starting at $s = 0$ when $t = 0$.*

1. If $s = t^4+5t^2+2t$, find the velocity and acceleration at $t = 1$.

2. If $s = \frac{1}{3}t^3-2t^2$, find
   (1) the velocity and position when the acceleration is 0;
   (2) the acceleration and position when the velocity is 12.

3. If $v = (t^3+2t)$ f.p.s., find
   (1) the average acceleration from 1 to 4 sec.;
   (2) the velocity when the acceleration is 14 f.p.s.$^2$

Given the distance-time formulae in Nos. 4, 5 find
   (1) the time when the velocity is zero;
   (2) the time when the acceleration is zero, and the velocity at this instant;
   (3) the distance from the starting-point, the velocity, and the acceleration when $t = 1$.

4. $s = t^3-75t$ (units: yd., min.).

5. $s = 4t+\frac{3}{2}t^2-\frac{1}{3}t^3$ (units: ft., sec.).

6. If $s = 5t+10t^2$, show that the acceleration of the body is constant and find its value.

7. Repeat No. 6 when $s = gt+\frac{1}{2}ht^2$, where $g$, $h$ are constants.

8. If $s = bt+\frac{1}{6}(ct^3)$, where $b$, $c$ are constants, show that the acceleration varies as $t$.

9. If $v = 6t^2-26t+11$, find the acceleration at both instants when the velocity is 3. Also find the velocity at the instant when the acceleration is zero.

10. A body starts from the origin and after $t$ sec. is $x$ ft. from O, where $x = 9t^2-t^3$. Find (i) the time and position when $v = 0$; (ii) the time and velocity when the acceleration is zero; (iii) the velocity with which the body passes through the origin again; (iv) the position and velocity when the acceleration is $-42$ f.p.s.$^2$

11 Discuss the motion completely when $s = t^3-3t$ and the start is at $t = 0$

12 The times at which various distances were run by H A Russell of Cornell University during a race of 200 yd are given in the following table †

| Distance (yd ) | 0 | 1 | 3 | 6 | 10 | 15 | 20 | 40 |
|---|---|---|---|---|---|---|---|---|
| Time (sec ) | 0 | 0 36 | 0 75 | 1 205 | 1 715 | 2 275 | 2 760 | 4 595 |

| 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 |
|---|---|---|---|---|---|---|---|
| 6 380 | 8 125 | 9 885 | 11 695 | 13 550 | 15 455 | 17 425 | 19 455 |

Draw the distance time graph from 0 to 2 sec Verify that some of the entries in the following table (the results of my calculations) are approximately correct

| Time (sec ) | 0 | 0 2 | 0 4 | 0 6 | 0 8 | 1 0 | 1 2 |
|---|---|---|---|---|---|---|---|
| Velocity (yd per sec ) | 0 | 3 0 | 4 3 | 5 4 | 6 3 | 7 0 | 7 6 |

Draw the velocity time graph from 0 to 1 2 sec

Find approximately the acceleration (*a*) at 0 4 sec, (*b*) at 1 0 sec When is the acceleration a maximum? Estimate the maximum acceleration as accurately as you can

Find the average acceleration for the first second

Show from the distance time table of values that the runner attains his greatest speed between 60 and 80 yd and that the average speed between these distances is approximately 23 44 m p h

Assuming that the error in the times is not more than $\pm\frac{1}{400}$ sec and that the distances are correct, show that all that can be said with certainty is that this speed lies between 23 4 and 23 5 m p h

EXAMPLE 2 A body moving in a straight line, starts with velocity 15 f p s and its acceleration after $t$ sec is $(12-6t)$ f p s² Find (i) the velocity and distance from the starting point in terms of the time, (ii) when and where its velocity is zero

*Solution* (i) If $v$ f p s is the velocity $t$ sec after the start,

$$\frac{dv}{dt} = 12-6t$$

Therefore $v = 12t-3t^2+c$, where $c$ is a constant

When $t = 0$, $v = 15$

Therefore $15 = c$ and $v = 15+12t-3t^2$ or

$$\frac{ds}{dt} = 15+12t-3t^2$$

Therefore $s = 15t+6t^2-t^3$, since $s = 0$ when $t = 0$

† A V Hill *Living Machinery* p 250

(ii) $v = 0$ when
$$15+12t-3t^2 = 0,$$
$$5+4t-t^2 = 0,$$
$$(5-t)(1+t) = 0.$$

The only positive root is 5.
Hence the velocity is zero after 5 sec.
Its position then is given by
$$s = 75+150-125 = 100.$$

The velocity is zero 5 sec. after the start and 100 ft. from the starting-point.

## Exercise 3.P

*The following questions refer to the motion of a body in a straight line.*

1. The velocity of a body after $t$ sec. is given by $v = 3(t^2+\frac{1}{4}t)$ f.p.s. How far does the body travel in the first 4 sec.?

2. A body starts from rest at $t = 0$ and moves with acceleration $(t+1)$ f.p.s.$^2$ where $t$ sec. is the time from the start. What is the velocity after 4 sec.? What is the distance travelled after 3 sec.?

3. A body starts with velocity 5 f.p.s. and after $t$ sec. its acceleration is $(4-t^2)$ f.p.s.$^2$ What is its velocity after 6 sec.?

4. A body starts from rest and moves so that $v = 24t(1-t)$ f.p.s. after $t$ sec. How far has it travelled when it is again at rest?

5. A body is thrown vertically upwards and after $t$ sec. its velocity is $v = (90-32t)$ f.p.s. What is its height after ($a$) 2 sec., ($b$) 5 sec.? After how long and with what velocity does it return to its starting-point?

6. At $t$ sec. $v = \{(t+5)/10\}$ f.p.s. How far must the body travel from its position at $t = 0$ to reach a speed of 2·5 f.p.s.?

7. At $t$ sec. $v = 3(1+t^2)$ f.p.s. Find the distance travelled when the acceleration is 12 f.p.s.$^2$ Find the velocity and acceleration at the start ($t = 0$).

8. A train starts with velocity 2 f.p.s. and moves with acceleration $\frac{1}{10}t$ f.p.s.$^2$, $t$ being the time in sec. from the start. After how long will the velocity be 7 f.p.s.? How far will the train travel in 12 sec.?

9. A body starts with velocity $\frac{2}{3}$ f.p.s. and after $t$ sec. its acceleration is $(1-t)$ f.p.s.$^2$ After how long will it be back at the starting-point, and what will be its velocity then?

10. A train starts from rest at a station and after $t$ min. its acceleration is $900t(2-t)$ yd. per min.$^2$ until this expression vanishes, when the acceleration remains zero until the train brakes for the next station. Find the maximum velocity of the train between the stations and the distance in which this velocity is attained.

11 A car passes a policeman and continues at the constant speed of 750 yd per min for 1 min Then its acceleration is $100(2-t)$ yd per min $^2$, where $t$ min is the time since passing the policeman What is its speed 2 min after passing the policeman?

12 A stone is thrown vertically downwards from the edge of a cliff with velocity 20 f p s Its acceleration is 32 f p s $^2$ Find the velocity with which it hits the water 4 sec later Also find the height of the cliff

13 A body starts with velocity $-72$ f p s and moves with acceleration $\frac{1}{4}t^2$ f p s $^2$, where $t$ is the time in sec from the start Find when the velocity is $+72$ f p s and the position of the body relative to the starting point then

14 At $t$ sec $v = 6t^2-22t+12$ Show that the body passes the starting point twice and find the velocity and acceleration on each occasion Show that it changes direction twice and find the distances from the starting point where this occurs

15 (a) A body starts with velocity $u$ and moves with constant acceleration $a$ Find the velocity and distance at time $t$ in terms of $u$ and $t$

(b) A body starts with velocity $u$ and moves with acceleration $kt$, where $k$ is a constant Find the velocity and distance at time $t$

## 3.15. Further families of curves

EXAMPLE 1 Find the equation of the curves for which

$$d^2y/dx^2 = 12x$$

*Solution* We have $dy/dx = 6x^2+a$, where $a$ is a constant Hence $y = 2x^3+ax+b$, where $b$ is another constant

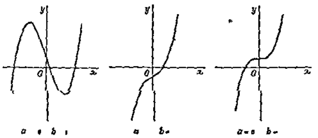



FIG 3 24

This equation contains two constants which may be given any values we like Each of the curves so obtained is such that the gradient of its gradient function is $12x$ The curves are therefore said to form a family This means that we widen the

definition of a family to include the case when the equation of the family contains any finite number of constants.

The gradient function of the family of curves obtained in Example 1 is $6x^2+a$. Hence when $a$ is negative there are two stationary values of $y$ and when $a$ is positive there are none. In both cases the point of inflexion is on $x = 0$.

The graphs of typical curves of the family are shown in Fig. 3.24.

EXAMPLE 2. Find the equation of the curve passing through (1, 0) and (−1, −2) for which $d^2y/dx^2 = 12x$.

*Solution.* As before, the equation of the family is $y = 2x^3+ax+b$.
As (1, 0) is on the curve $0 = 2+a+b$.
As (−1, −2) is on the curve $-2 = -2-a+b$.
Solving these equations for $a$ and $b$ we find $a = -1$, $b = -1$.
The equation of the curve is $y = 2x^3-x-1$.

Note that when the equation of the family contains two constants, two conditions are necessary to enable us to find a particular curve.

## EXERCISE 3.Q

1. A family of curves is such that $d^2y/dx^2 = x^2$. Find the equation of the family. Find the equation of the curve of the family which passes through (0, 1) and $(-1, \frac{1}{12})$.

2. Find the equation of the family of curves for which $d^2y/dx^2 = 2$. Sketch the curves for different values of the constants.

3. Find the equation of the family of curves for which

$$\frac{d^2y}{dx^2} = \frac{x-1}{x^4}.$$

4. A curve is such that $d^2y/dx^2 = 1+x$ and the points (0, 2) and (1, 4) are on it. Find the equation of the curve.

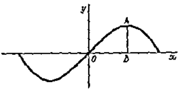
5. A curve is such that $d^2y/dx^2+2x = 0$, and it passes through the origin and has gradient $\frac{1}{3}$ there. Find its equation.

6. A curve is such that $d^2y/dx^2 = 2-3x$. It passes through (−1, 2), and when $x = 2$ the gradient is $-\frac{8}{2}$. Find the equation of the curve.

7. A curve is such that $d^2y/dx^2 = x$; it passes through the origin, and the equation of the tangent at the origin is $y = x$. Find the equation of the curve and the equation of its tangent at the point where $x = 1$.

8. Find the equation of the curve which has $d^2y/dx^2 = 6(1+10x^4)$ and which passes through (1, 10) and (−1, 4).

9 A manufacturer makes an $S$ bracket as shown in the sketch




He wants it to bend one way when $x$ is positive and the opposite way when $x$ is negative He therefore takes $d^2y/dx^2 = -\frac{3}{2}x$ He makes the curve pass through (0, 0) and (2, 0) Find the equation of the curve and show that it passes through $(-2\ 0)$ If $AB$ is the maximum height of the curve, show that $AB = 4/(3\sqrt{3})$

10 A curve has $d^2y/dx^2 = 9x^2$ and has a stationary value of $-10$ when $x = 2$ Find its equation and whether the stationary value is a maximum or minimum

## 3.16. Differential Equations

Equations such as

$$\frac{dy}{dx} = 6x \tag{1}$$

$$(\text{or } dy = 6dx)$$

$$\frac{d^2y}{dx^2} = x^2, \tag{2}$$

are called *differential equations*

We know from previous work that we can find the equation of a family of curves such that the gradient function of each curve satisfies (1) In fact the equation of this family is $y = 3x^2+c$, where $c$ is an arbitrary constant

We call this family of curves a *solution* of the differential equation (1)

A solution of the differential equation (2) is found as follows

$$\frac{d^2y}{dx^2} = x^2,$$

$$\frac{dy}{dx} = \frac{x^3}{3}+c,$$

$$y = \frac{x^4}{12}+cx+d,$$

where $c$ and $d$ are constants For all values of $c$ and $d$ this family of curves has a gradient of the gradient which satisfies (2), and it is called a *solution* of the differential equation (2)

Note that differential equations which contain $dy/dx$ but not $d^2y/dx^2$ have a solution which contains one constant, while differential equations which contain $d^2y/dx^2$ have a solution which contains two constants

EXAMPLE. Solve the differential equation

$$x^2\frac{dy}{dx}+2 = 0.$$

*Solution.* Write the equation in the form $dy/dx = -2/x^2$.
Hence $y = 2/x+c$, where $c$ is a constant.
This is the required solution.

## EXERCISE 3.R

1–8. Solve the following differential equations:

1. $dy/dx = 6x^2-16x+4$.
2. $dy = (x+2x^2)dx$.
3. $dy/dt = 2t^3-\frac{3}{2}t^2+1$.
4. $dy/dx = x^2-1/x^4$.
5. $d^2y/dx^2 = 4x$.
6. $d^2h/dt^2 = 2t(t^2+1)$.
7. $dv/dt = 1-1/t^2$.
8. $d^2s/dt^2 = 2/t^3-24/t^4$.

9–13. Solve the following differential equations subject to the given conditions:

9. $dA/dx = 30x^2(x^2-1)$, $A = 0$ when $x = 1$.
10. $dv/dh = h^2+1/h^2$, $v = \frac{25}{6}$ when $h = 2$.
11. $d^2s/dt^2 = t(t-2)$, $ds/dt = 4$ when $t = 3$, and $s = 0$ when $t = 0$.
12. $d^2y/dx^2 = 4x-5$, $dy/dx = 1$ when $x = 2$, and $y = 2$ when $x = 3$.
13. $d^2\theta/dt^2 = 1-\frac{1}{6}t$, $d\theta/dt = \frac{1}{2}$ when $t = 2$, and $\theta = 0$ when $t = 0$.
14. Find the equation of the curve whose gradient function is $x$ times the gradient function of $y = x^3$ and which passes through the origin.
15. Show that the family of curves for which $dy = a\,dx$, where $a$ is a constant, consists of parallel straight lines.
16. Find the equation of the curves given by $3dy = 4x^2\,dx$. Find the equation of the curve through (2, 3).
17. Find the equation of the curve through $(2, \frac{5}{3})$ for which $3dy+x^3\,dx = 0$.
18. Find the solution of $d^2y/dx^2 = 10/x^3$ for which $y = 8$ when $x = 1$, and $y = 12$ when $x = 5$.
19. Find the solution of $dy/dx = 16x^2-1/x^2$ if the minimum value of $y$ is $\frac{2}{3}$. Find the maximum value of $y$.
20. Show that the family defined by $d^2y/dx^2 = 0$ contains all the straight lines in the plane of the axes except those parallel to the $y$-axis.
21. The gradient of a road (defined as in graphs) steadily increases from 1 in 80 at $A$ to 1 in 16 at $B$, 400 ft. horizontally from $A$, so that the gradient $x$ ft. from $A$ is $\frac{1}{80}+kx$ with a suitable value of $k$. Determine $k$, and if $A$ is 100 ft. above sea-level, find the height of $B$.

3 17 We have seen that the solution of a differential equation is a family of curves Conversely from the equation of a family we can obtain *a differential equation characteristic of the family*

EXAMPLE 1 Find the differential equation of the family of curves $y = ax^3$

*Solution* We have $$\frac{dy}{dx} = 3ax^2$$

Eliminating $a$, $$x\frac{dy}{dx} = 3ax^3 = 3y$$

The differential equation of the family is

$$x\frac{dy}{dx} = 3y$$

EXAMPLE 2 Find the differential equation of the family $y = ax^3+b$

*Solution* We have $$\frac{dy}{dx} = 3ax^2,$$

$$\frac{d^2y}{dx^2} = 6ax$$

Hence $x\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$ is the required differential equation.

EXAMPLE 3 Find the differential equation of the family

$$y = ax^3+bx \tag{1}$$

*Solution* We have $$\frac{dy}{dx} = 3ax^2+b, \tag{2}$$

$$\frac{d^2y}{dx^2} = 6ax \tag{3}$$

Eliminating $b$ from (1) and (2),

$$y = ax^3+x\left(\frac{dy}{dx}-3ax^2\right)$$

$$= x\frac{dy}{dx}-2ax^3 \tag{4}$$

Eliminating $a$ from (3) and (4)

$$y = x\frac{dy}{dx}-\frac{x^2}{3}\frac{d^2y}{dx^2},$$

or $$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y = 0$$

Note that just as we found that a differential equation involving $d^2y/dx^2$ led to a solution containing two arbitrary constants, so we find it necessary to find both $dy/dx$ and $d^2y/dx^2$ when the equation of the family contains two constants

## Exercise 3.S

1–12. Find the differential equations of the following families of curves.

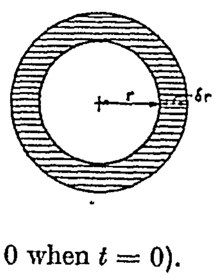
*Letters other than $x$ and $y$ are the constants defining the individual curves of the families.*

1. $y = ax^2$.
2. $y = mx+k$.
3. $y = mx+1$.
4. $y = ax^2+c$.
5. $2y = c^2/x$.
6. $y = ax^2+bx$.
7. $y = x^3+ax^2$.
8. $y = ax+b/x$.
9. The lines through the origin.
10. The lines parallel to $y = 2x$.
11. The lines making equal intercepts (taking account of signs) on the axes.
12. The lines through the point $(-1, 2)$.
13. The differential equation of the family of circles whose centre is the origin is $$dy/dx . y/x = -1.$$

Show that this expresses the geometrical fact that the tangent at any point is perpendicular to the radius to that point.

## Miscellaneous Exercise 3.X

1. Sketch the curve $y = 3x^2$ from $x = 0$ to $x = 2$. At $x = 1$ choose a value for $\delta x$ and show on the figure $\delta y$ and $dy$.
2. Find the equation of the normal to $y = x^3-3x$ at $(2, 2)$.
3. If $y = x^2-x$, find the stationary value of $y$ and decide whether it is a maximum or minimum.
4. If $s = 2t^3-4t$, find formulae for $v$ (velocity) and $a$ (acceleration) in terms of $t$.
5. Show that the area of a circular ring of inner radius $r$ and width $\delta r$ is approximately $2\pi r\, \delta r$.




6. If $y = x^2(7-2x^5)$, find $d^2y/dx^2$. Show that $y$ has a stationary value when $x = 1$.
Find the stationary value and decide whether it is a maximum or a minimum.
7. If $v = 4t+8$, find $a$ and $s$ in terms of $t$ ($s = 0$ when $t = 0$).
8. Find the point of inflexion on $y = x^2(x+6)$.
9. The gradient function of a curve is $x^2(3-10x^2)$ and the curve makes an intercept of 3 on the $y$-axis. Find the equation of the curve.

10 The pages of a book are to have margins of $\frac{1}{2}$ in at the sides of the print and of 1 in at the top and bottom If the area of the print is to be 32 sq in, find the dimensions of a page of minimum area [Hint let $x$ in denote the width of the *print*]

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11 If $y = x^3$, find $dy$ in terms of $x$ and $dx$ Evaluate $(2\ 1)^3$ approximately

12 Find the maximum value of $y = 32x - x^4$

13 Find the coordinates of the point of inflexion on

$$6y = 2x + 6x^2 + x^3$$


14 A church window consists of a semicircle surmounting a rectangle If the perimeter of the window is fixed, show that its area is a maximum when $h = r$

15 If $s = 10 + 5t - 16t^2$, find formulae for $v$ and $a$

16 If $a = 10$, and $v = u$ when $t = 0$, $s = 0$ when $t = 0$ find formulae for $v$ and $s$

17 Sketch the curve $y = x^2 - 2/x$ near the point where $x = 1$, showing the tangent there and how the curve bends

18 At all points of a curve, $d^2y/dx^2 = 1/x^4$ If the curve passes through ($\frac{1}{2}$ 2) and (1, 1), find its equation

19 Find the differential equation of the family of curves $y = ax + 1/x$

20 The tangent at the point of inflexion of the curve whose gradient function is $2x - x^2$ is $y = x + 1$ Find the equation of the curve

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21 The lift on an aeroplane's wing is $0\ 2v^2$ lb wt, where $v$ f p s is its speed through the air Find approximately the increase in lift if the airspeed increases from 200 to 220 f p s

22 Find the equations of the normals at the points where $2y + 1 = x^2$ meets the $x$ axis

23 Find the maximum and minimum values of $y = x^3 - 3x$

24 If $s = 6t^4 + 100t$, find the velocity and acceleration when $t = 2$

25 Find the equation of the curve for which $d^2y/dx^2 = 2 - 9x^2$, if it has gradient 1 at the point (1 1)

26 Compare the graphs of $y = x^3$ and $y = x^3 + x$ showing that the addition of the term $x$ changes the gradient but not the way the curve bends Show that their point of intersection is a point of inflexion on each curve

27 A manufacturer wishes to make, as economically as possible, an aluminium saucepan with a tin lid Assuming that the volume is to be $36\pi$ cu in and that the sheet aluminium and sheet tin used

cost 0·3 and 0·1 pence per sq. in. respectively, find the dimensions of the cheapest saucepan and show that the cost of the materials is a little less than 2*s*. 10*d*.

28. Find the differential equation of the family of curves $4ay = x^2$.

29. Sketch $y = 1/x^2$. Find the equations of its tangents at the points where it meets $2xy = x-1$.

30. The gradient function of a family of curves is $ax+1$. Find the equation of the family and its differential equation.

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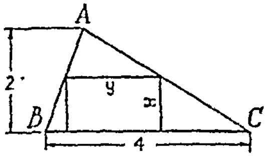
31. The triangle $ABC$ has height 2 in. and base 4 in. A rectangle is inscribed in the triangle with one side on $BC$. If $x$, $y$ in. are the height and length of this rectangle, show that $y = 4-2x$ and determine the values of $x$ and $y$ when the area of the rectangle is a maximum.




Verify that the area of the maximum rectangle is one-half the area of the triangle.

32. The slope $\alpha$ of a curve at any point $(x, y)$ is given by $\tan\alpha = x/10$. Find the equation of the curve if it passes through $(4, -\frac{1}{5})$.

33. Evaluate $3{\cdot}2^4$ approximately, giving the answer to 3 sig. fig.

34. Find the gradient of $1/x$ when $x = a$. Find the equation of the tangent at $P \equiv (a, 1/a)$ to $y = 1/x$. If this tangent meets the $x$-axis at $T$ and the foot of the perpendicular from $P$ to the $x$-axis is $N$, show that $OT = 2ON$, where $O$ is the origin.

35. Sketch the curve $y = 2+1/x$ in the neighbourhood of $x = 1$, showing the tangent at $x = 1$ and the way the curve bends.

36. If $a = 12t$, $v = 10$ when $t = 1$, and $s = 0$ when $x = 0$, find formulae for $v$ and $s$.

37. If $v = (3t^2+1)$ f.p.s., find the acceleration when $v = 4$ and the distance in which the velocity increases from 4 to 76 f.p.s.;

38. If $y = 2x^3-6x^2$, find the gradient of the tangent at the point of inflexion.

39. A ship springs a leak and the depth of water $x$ ft. in the hold after $t$ hrs. obeys the law $dx/dt = 4t$. In what time will the depth increase by 18 ft. ?

40. If the horse-power, $H$, required to propel a certain ship at $v$ knots is given by $H = 2{\cdot}75v^3+12000$ find approximately the increase in horse-power necessary to increase the speed from 20 to 21 knots.

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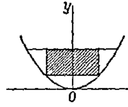
41 Find the gradient function of $\pi x^2(1-3/x^4)$




42 Rectangles are inscribed in the part of the curve $4y = x^2$ bounded by $y = 3$ as shown in the sketch Find the dimensions of the rectangle with maximum area

43 The rate of change of $y$ with $x$ is $(1-x)^2$ and $y = \frac{1}{3}$ when $x = 1$ Find $y$ in terms of $x$

44 Find the limits of (a) $1+h+h^2$, (b) $(1+h)/(2+h)$, (c) $2x^2+xh+h^2$, when $h \to 0$

45 Solve the differential equation $d^2y/dx^2 = 1-1/x^3$ if $y = 1$ and $dy/dx = \frac{1}{2}$ when $x = 1$

46 Find the point of inflexion on $y = 9x^2-12x-2x^3$ Also find the maximum and minimum values

47 Show that the gradient of $\frac{1}{3}x^3+x(1-x)$ is never negative Show that $x^3 > 3x(x-1)$ when $x > 0$

48 If $y = 1/x^3$, $x = 1$, $dx = 0\,02$, find $dy$ Illustrate by a sketch What is an approximation to $1/(1\,02)^3$ ?

49 If the velocity of a body moving in a straight line is $(2+\frac{1}{10}t^2)$ f p s at $t$ sec after the start, find the acceleration and distance travelled 6 sec after the start

50 Sketch the curve $y = x^5+4/x$ and its tangent in the neighbourhood of the point (1, 5)

## MISCELLANEOUS EXERCISE 3 Y

1 Find the gradient functions of the following (a) $(1+x^2)(1-\frac{1}{2}x^2)$, (b) $2\pi x(1-4/x^2)$, (c) $(a^2-x^2)/x^4$ ($a$ is a constant)

2 Find the equations of the family of curves whose gradient function is (a) $(3+5x^2)^2$, (b) $1/x^2-2+3x^2$, (c) $(x^2-1)/x^6$

3 Find the equations of the tangent and normal at $(\frac{1}{2}, 2\frac{1}{2})$ to $y = x+1/x$ Show that the tangent does not meet the curve in any other point but that the normal meets it again in one other point, and find the coordinates of this point

4 Find the equation of the tangent at the point where $x = \frac{3}{2}$ on $y = x^2-4x$ Find the $x$ coordinate of the point where the normal is parallel to this tangent, and find the equation of this normal

5 Find the equation of the tangent at the point $P$, where $x = a$, on $y = 1/x^2$ If this tangent meets the $x$ axis at $T$ and the foot of the perpendicular from $P$ to the $x$ axis is $N$, show that $NT = \frac{1}{2}a$

6 $4y = x^2(x-6)$ Find the maximum and minimum values and the point of inflexion Sketch the curve

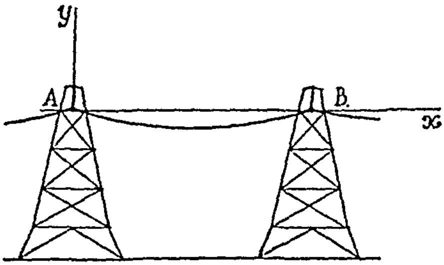
7. Find the maximum and minimum values of $y = 2x^2 - x^4$. Has the curve any symmetry? Sketch the curve.

8. $2y = x(12 - x^2)$. Find the turning-points and the point of inflexion, and sketch the curve.

9. A body travels in a straight line and its distance, $s$ cm., from the start after $t$ sec. is given by $s = 6t^2 - t^3$. Find the maximum velocity of the body. Find the greatest positive distance of the body from its starting-point. What is the average velocity of the body during the first 3 sec. of its motion?

10. The velocity of a body $t$ min. after the start is $v$ metres per min., where $v = t^2 + 2$.

(i) Find the distance travelled by the body during the fifth minute.
(ii) Find the average velocity during the first two minutes.
(iii) Find the acceleration when $t = 3$.




11. The equation of one span of a copper high-tension electric cable referred to horizontal and vertical axes through $A$ may be taken as $y = \{x(x-1000)\}/6250$ [units: ft.] when it is attached at $A$ and $B$ to pylons 1000 ft. apart.

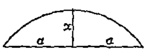
(i) Find the sag of the span (i.e. the greatest vertical distance of the wire below $AB$).
(ii) Find the angle between the two spans of wire meeting at $A$, assuming that they are of the same shape.
(iii) The corresponding equation for an aluminium wire is of the form $y = kx(x-1000)$, but the sag is only half the sag for a copper wire. Find $k$ and the angle between adjacent spans.

12. Find the least value of the sum of the square of a number and the square of its reciprocal.

13. The product of two positive numbers is 100. Show how to choose the numbers so that their sum has the smallest possible value, and find this value.

14. The sum of two positive numbers is $a$, a constant. Show how to choose the numbers so that their product is a maximum, and find the maximum value.

15 Circular arcs are drawn on a fixed chord of length $2a$ If the offset at the middle of the chord is $x$, find a formula for the radius of the circle in terms of $x$ Find the least possible value of the radius and explain the result geometrically




16 A rectangular box of height $y$ in has a square base of side $x$ in Show that the volume $v$ cu in and the surface area $A$ sq in are given by $v = x^2y$ and $A = 2x^2+4xy$

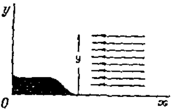
(a) If $v = 64$, find the values of $x$ and $y$ which make $A$ a minimum

(b) If $A = 150$, find the maximum volume which the box can have

17 A straight line 10 in long is drawn on a sheet of paper and you are asked to draw a right angled triangle which has this line as hypotenuse Show that the area of any right angled triangle you draw cannot exceed 25 sq in [Consider the square of the area ]

18 A formula for the height to which a jet of water issuing from the nozzle of a hose pipe rises is $h = 2{\cdot}304p - 0{\cdot}0072(p^2/d)$, where $h$ ft is the height of the jet, $d$ in is the diameter of the nozzle, $p$ lb per sq in is the pressure of the water in the nozzle Find the pressure at which the height of the jet is a maximum for a nozzle of $\frac{3}{4}$ in diameter Find the rate of increase of the height of the jet with the pressure of the water (a) when $p = 50$, (b) when $p = 100$

19 I found in a book on meteorology† the following information about the sea breeze The barometric pressure, $B$ mm of mercury, and the temperature $T°$ C of the air depend upon $x$, the origin being on the land and the positive $x$ axis stretching out to sea at right angles to the shore line The height, $y$ metres, to which the breeze reaches is given by




$$y = -30\frac{\delta B}{\delta T}\frac{(273+T)^2}{B},$$

where $\delta B$ and $\delta T$ are the increments in pressure and temperature as we pass from the land to the sea Calculate to the nearest metre the height of the sea breeze on a day when

| | *Over land* | *Over sea* |
|---|---|---|
| B = | 760 mm | 760·5 mm |
| T = | 27° C | 22° C |

20 Calculate approximately

(a) $(81{\cdot}5)^3$ (to the nearest thousand)

(b) $1000/(9{\cdot}9)^2$ (to one decimal place)

† W J Humphreys, *The Physics of the Air*

21. In 1936 the scheduled time of the *Flying Scotsman*, which had been 7 hr. 30 min. for the 392 miles from King's Cross to Edinburgh, was reduced by 15 min. Find approximately the increase in the average speed.

22. A 440-yd. race is timed by three timekeepers and the watches read $55\frac{4}{5}$, 56, and $56\frac{1}{5}$ sec. If the mean of these times is used to calculate the speed of the runner, give the speed with an estimate of its accuracy. [Assume that the exact time for the race is between the greatest and least of the three given times.]

23. The distance $s$ ft. in which a certain train reaches a speed $v$ m.p.h. from rest is given by the formula $s = 41v^2/32$. Find approximately the distance in which the speed increases from 40 to 45 m.p.h. (Give the answer in feet to 2 sig. fig.)

24. Sketch the graphs of

(i) $y = x^3$;  (ii) $y = x^4$;  (iii) $2y = x^3+x^4$.

Find the stationary values of (iii) and determine whether each is a maximum, minimum, or point of inflexion.

25. Find the equation of the family of curves for which

$$d^2y/dx^2 = 2(1-4x).$$

One curve passes through (0, 2) and its tangent there is parallel to the $x$-axis. Find its equation.

26. The curve $y = x^2+ax+b$ has a stationary value of 1 at $x = 2$. Determine the constants $a$ and $b$. Is the stationary value a maximum or minimum ?

27. Find the equation of the curve with gradient function $a-x$, if it has a stationary value of 7 at $x = 2$. ($a$ is a constant to be determined.) Is the stationary value a maximum or minimum ?

28. The gradient function of a curve is $b+ax-12x^2$; there is a point of inflexion when $x = -\frac{1}{4}$ and a stationary value when $x = -1$; when $x = 1$, $y = 0$. Determine $a$ and $b$ and find the equation of the curve. Also find the stationary value and whether it is a maximum or minimum.

29. Find the equation of the normal at $A \equiv (\frac{1}{10}, \frac{1}{20})$ to $y = 5x^2$. If this normal meets the curve again at $B$, show that the length of $AB$ is $2\sqrt{2}/5$.

30. The velocity, $v$ yd. per min., of a body at $t$ min. is given by $v = 4t^2-t^3$. Find the acceleration at $t = 2$ and the distance travelled from $t = 0$ to $t = 2$. Find the average velocity during this 2 min. Find the maximum velocity.

31. A body starts at $t = 0$ with velocity 2 f.p.s. and moves with acceleration $(2-3t)$ f.p.s.$^2$ At what distance from the start does it begin to return to the starting-point ?

32 An express travelling at 5000 f p min is approaching a junction where there is a speed limit of 25 m p h We begin to consider the motion when the driver puts on his brakes It is convenient to define this instant as $t = -1$ min because then the acceleration of the train can be given in the form $a = 12000t^3$ from $t = -1$ to $t = +1$ Verify that the train retards from $t = -1$ to $t = 0$ and accelerates from $t = 0$ to $t = 1$ The junction is passed at $t = 0$ What is the speed then? Show that at $t = 1$ it is again travelling at 5000 f p min, and find the distance travelled from $t = -1$ What is the average speed during the check?

33 Find the equation of the chord joining $A \equiv (-1, 4)$ and $B \equiv (3, 36)$ on $y = 4x^2$ and find the coordinates of its middle point $M$ A parallel to the $y$ axis through $M$ meets the curve at $P$ Show that the tangent at $P$ is parallel to $AB$

34 Find the gradient functions of

(i) $y = (x^2-a)^2$, (ii) $y = \{\pi(a+bx)\}/x^4$, (iii) $s = k(t^2-1/t)^2$, where $a$, $b$, $k$ denote constants Find the gradient of the gradient function when $y = (1-x^2)^2/x^3$

35 Find the differential equation of the family of curves $y = ax^2+bx+1$

36 Find the equation of the family of curves for which

$$d^2y/dx^2 = 1+3/x^4$$

Find the equation of the curve of the family (a) which passes through the point (1, 1) and has gradient 1 there, (b) which passes through (−1, 2), (1, 0), (c) which is symmetrical with respect to the $y$ axis and passes through (1, 1)

Do any of these curves pass through the origin?

37 The horse power, $H$, transmitted by a cotton driving rope of diameter $d$ in moving at speed $v$ ft per min. is

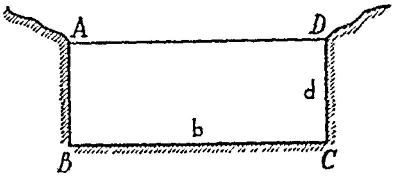
$$H = \frac{d^2v}{33000}\left(160-\frac{v^2}{425000}\right)$$

At what speed, to 2 sig fig, is the horse power transmitted by a given rope a maximum? Should driving ropes of different diameter be driven at the same speed to transmit maximum horse power?

38 The total air and frictional resistance to the motion of a certain locomotive travelling at $v$ m p h on a level track amounts to $R$ lb wt, where $R = 425+4\ 5v+0\ 2v^2$ Find, to the nearest lb wt, the change in the resistance when the velocity changes (a) from 50 to 51 5 m p h, (b) from 30 to 28 m p h

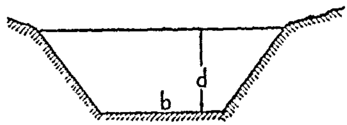
39 An engineer, designing a rectangular channel to carry water, has to determine the shape of the channel which will carry most water for a given amount of excavation

(*a*) Given that the quantity of water carried is a maximum when the wetted perimeter of the cross-section ($AB+BC+CD$) is a minimum, and that the engineer fixes the area of the cross-section at 72 sq. ft., find the values which he should assign to $b$ and $d$.




(*b*) More generally, show that if the area of cross-section is fixed, the wetted perimeter is a minimum when $b = 2d$.

(*c*) If the sides of the channel slope with gradient $\frac{4}{3}$ and the area of the cross-section is fixed at 112 sq. ft., find what the depth and the breadth at the bottom must be in order that the wetted perimeter may be a minimum.




40. The strength of a rectangular wooden beam of breadth $b$ and depth $d$, as measured by the greatest load it can carry without breaking, is given by $s = kbd^2$, where $k$ is a constant. Find the dimensions of the strongest beam which can be cut from a circular tree trunk of diameter 3 ft.

Show that, whatever the diameter of the trunk, the ratio of the depth to the breadth of the strongest beam is $\sqrt{2} : 1$.

## Miscellaneous Exercise 3.Z

1. Find the gradient functions of: (i) $2x^3-7x^2+5$, (ii) $ax^3-b/x^4$, (iii) $(x^5-1/x^5)^2$, (iv) $(at-b)/t^5$, where $a$, $b$ are constants.

2. Give the equations of the families of curves which have the following gradient functions: (i) $6x^2-5x+4$, (ii) $(x^4-a)/x^2$, (iii) $(x-a)^3$, where $a$ is a constant.

3. Find the equations of the tangent and normal at $(1, -1)$ to $y = x^3-2x$. Find the equation of a parallel tangent and the equation of the corresponding normal. Explain the result with a figure.

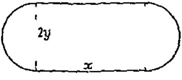
4. Show that the curves of the family $y = 27x(2x-1)^2+k$ all have their stationary values at the same value of $x$. Determine $k$ for the curve which has a minimum value of $-2$ and determine the maximum value for this curve.

5. Find the equations of the tangents at the points of inflexion of $y = \frac{1}{4}x^4+x^3-x$.

6. Find the coordinates of the point on $8y = x^2$ at which the normal is parallel to the tangent at the point $(2, \frac{1}{2})$. Find the equation of this normal and the coordinates of the point in which it meets the curve again.

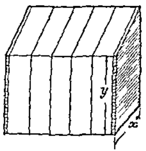
7 $y = 2x^3 - 12x^2 - 30x + 100$ Find (i) the maximum and minimum values of $y$, (ii) the minimum value of $dy/dx$

8 The horse power, $P$, transmitted by a certain leather belt running over two pulleys with a speed of $v$ ft per min is given by $P = 8 \times 10^{-3} v(1 - 10^{-8} v^2)$ Find the velocity at which the horse power transmitted is greatest




Ex 3 Z 9

9 A cylindrical boiler has hemispherical ends and its surface area is $A$, a constant Show that the volume is a maximum when $x = 0$




Ex 3 Z 10

10 A man builds a loggia against the wall of his house The ends of the loggia are to be brick and the roof and front are to be glass The gradient of the roof is $\frac{3}{4}$ Show that if the width of the loggia and the area of the glass are given, the volume is a maximum when $x = 2y$

11 A body moves in a straight line so that the distance travelled, $s$ ft, at time $t$ sec after the start is given by $s = 96t - 2t^3$

(i) With what velocity does the body start?

(ii) Show that the acceleration varies as $t$ What is its sense?

(iii) When is the velocity instantaneously zero? Find the acceleration and the distance of the body from the starting point at this instant

12 $y = x^4 + ax^2 + b$, where $a$, $b$ are numbers Choose $a$ and $b$ so that $y$ has a maximum value of 4 when $x = 0$ and a minimum value when $x = 2$ Find the minimum value and show that $y$ has another equal minimum

13 $y = x^3 + ax^2 + bx + c$ Choose $a$, $b$, $c$ so that $y$ has a stationary value of $-1\frac{1}{2}$ at $x = 1\frac{1}{2}$ and a point of inflexion at $x = \frac{1}{2}$ Is the stationary value a maximum or minimum? Show that there is another stationary value and find the value of $x$ at which it occurs

14 The breaking load of solid rectangular mild steel beams is $L = 14(bd^2/s)$, where $L$ is the load in tons, and $b$, $d$, $s$ are the breadth, depth, and span in inches A beam is 2 in broad, 5 in deep and its span is 56 in Find, to the nearest hundredweight, the change in the breaking load if (i) $d$ is increased by 0 1 in, (ii) $s$ is increased by 1 in

15. Normally the density of the air decreases with height. If $\rho$ is the density and $h$ is the height above ground its rate of change with height is given by

$$\frac{d\rho}{dh} = -\frac{p}{(T+273)^2R}\left(\frac{dT}{dh}+0{\cdot}034\right),$$

where $h$ is in metres, $p$ is the pressure, $T$ is the temperature in degrees centigrade, and $R$ is a constant. If the density happens to be constant for a range of height, the air will be on the point of rising and vertical air currents will be set up.

When this happens

(a) What is the value of $d\rho/dh$?

(b) What value does the equation then give for the rate of change of temperature with height?

(c) Does the temperature increase or decrease as height increases?

(d) Express the rate of change of temperature in ° C. per km.

16. Newton's law, for the rate of cooling of a body which is at a higher temperature than its surroundings, is $dT/dt = -k(T-T_0)$, where $T$ is the temperature of the body, $T_0$ is the temperature of its surroundings, and $t$ is the time.

(a) State the law in words.

(b) The temperature of a cup of tea is 80° C. and is found to be falling at 2° C. per min. in a room at 15° C. Find $k$. Find approximately how long it will take the temperature to fall (i) from 60 to 59° C., (ii) from 28 to 27° C.

17. The starting force in lb. wt. which can be exerted by a 3-cylinder locomotive is given by the formula

$$T = \frac{1{\cdot}28\,d^2Pl}{D},$$

where $T$ lb. wt. = starting force,
$d$ in. = diameter of cylinder,
$l$ in. = length of cylinder,
$P$ lb. per sq. in. = boiler pressure,
$D$ in. = diameter of the driving wheels.

For a British Railways (Southern Region) 'Schools' class locomotive, $d = 16\frac{1}{2}$, $l = 26$, $P = 220$, $D = 79$. Find, to two sig. fig., the effect upon the starting force of making each of the following changes in turn in the design of this type of locomotive,

(a) an increase of $\frac{1}{2}$ in. in the diameter of the cylinder,

(b) an increase of 1 in. in the diameter of the driving wheels.

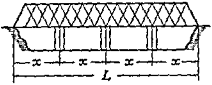
18. A body starts with velocity $u$ and moves so that after $t$ sec. its acceleration is $6t^2$. Show that its average velocity in the first $T$ sec. is its actual velocity after $\sqrt[3]{2}\,T/2$ sec.

19 If $y = 1/x$, show that

$$\delta y - dy = \delta x^2/\{x^2(x+\delta x)\}$$

and sketch the curve and its tangent

$$(a) \text{ near } x = 1, \ (b) \text{ near } x = -1$$




20 An engineer has to design a railway bridge to cross a river of width $L$ He must first decide how many spans to use Unless there are other considerations (e g unsuitable river bottom or minimum width for safe navigation) he will naturally choose the number of spans so that the cost of the bridge is least He estimates the cost of each pier to be £$P$ and the cost of each span to be $G = £ax^2$, where $a$ is a constant and $x$ is the length of the span

Assuming an unknown number of spans, $n$, find a formula for the total cost and express it in terms of $n$ and the constants Show that the cost of the bridge is least when $P = G$ Assuming $P = £1000$, $L = 300$ ft, $a = \frac{5}{8}$ find $x$ and $n$

21 Show that the differential equation of the family of curves $xy = c^2$ may be written in the form $y/x + dy/dx = 0$ If $P$ is any point on one of the curves, what does this equation show about the directions of $OP$ (where $O$ is the origin) and the tangent at $P$?

$P$, $Q$ are points on different curves of the family such that the tangents at $P$ and $Q$ are parallel Show that $PQ$ passes through the origin

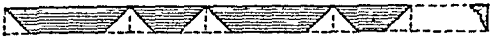
Find the equations of the normals to $xy = c^2$ which are parallel to $y = 4x+1$

22 The family of curves for which $d^2y/dx^2 = x-1$ clearly have their points of inflexion on the line $x = 1$ Find the equation of the family if the tangents at their points of inflexion are parallel to the $x$ axis and show that the curves have no maxima and minima Find the equation of the curve which passes through the origin

23 Find the point of inflexion on the curve $y = x^3-3x^2+9x-9$ Sketch the curve

24 A farmer has a certain number of hurdles available and he decides to make two sheep folds with them, one a square fold and the other circular Show that the total area enclosed by the hurdles is a maximum when a side of the square equals the diameter of the circle

25. Find the stationary values of $y = x^3(x^2-5x+5)$ and distinguish them.



26. A picture frame is made from a moulding $a$ in. wide, the ends of the four lengths of the frame being cut at an angle of 45° to the length of the moulding. Show that the length of moulding required to frame a picture of given area is least when the picture is a square.

## IV

## AREAS

**4.1.** IN elementary geometry we learn to calculate the area of a *figure bounded by straight lines The proofs of the formulae for* the areas of such figures (e g parallelogram, triangle) are based on the fact that the area of a rectangle is found by multiplying together its length and breadth One figure bounded by a curved line—the circle—is considered in elementary work, but the formula for its area is not proved in the sense that the formula for the area of a triangle is proved

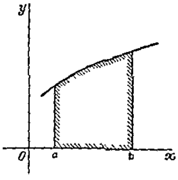



FIG 4 1

The problem we take up in this chapter is the calculation of an area when part, at least, of its boundary is curved We aim at making this calculation depend upon the for mula for the area of a rectangle When this problem was first solved in the middle of the seventeenth century it was shown by a group of mathematicians, of whom Sir Isaac Newton (1642–1727) was the chief, that the solution depends upon gradients

### 4 2 The area under a given curve

Consider a figure bounded by three straight lines and one curved line It is supposed that the $(x, y)$ equation of the curved line is known The remaining boundaries are $x = a$, $x = b$, $y = 0$, and $y$ is positive between $x = a$ and $x = b$ An area of this type is described as *the area under the curve from* $x = a$ *to* $x = b$ To evaluate this area we first consider an area under the curve between $x = a$ and a variable line parallel to the $y$ axis at a variable dis tance, $x$, from it We do this in order to find out how the area changes as one of its boundaries moves

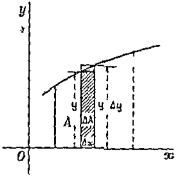



FIG 4 2

Denoting the area under the curve from $x = a$ to $x = x$ by $A$ we note that $A$ depends upon the position of the variable boundary, i.e. upon $x$. If $x$ is increased by $\Delta x$, the increase in $A$ (the shaded area in Fig. 4.2) is properly denoted by $\Delta A$. The figure also shows two rectangles of areas $y\Delta x$ and $(y+\Delta y)\Delta x$ and makes it clear that the area $\Delta A$ is between these two areas in magnitude. With this figure, drawn for a rising curve, we have

$$y\Delta x < \Delta A < (y+\Delta y)\Delta x.$$

For a falling curve we should have

$$y\Delta x > \Delta A > (y+\Delta y)\Delta x.$$

[Draw your own figure and verify this.]

In either case $\Delta A$ is between $y\Delta x$ and $(y+\Delta y)\Delta x$.

Therefore $\Delta A/\Delta x$ is between $y$ and $y+\Delta y$.

Let $\Delta x \to 0$. Since $A$ depends upon $x$, $\Delta A/\Delta x \to dA/dx$.

Also $y+\Delta y \to y$ provided the curve has no breaks in it (i.e. no point such as $x = 0$ on $y = 1/x$ must lie on the curved boundary).

Therefore $$\frac{dA}{dx} = y.$$

This is the result discovered by Newton.

EXAMPLE. Find the area under the curve $y = x^2$ from $x = 1$ to $x = 4$.

*Solution.* Draw a sketch. Let $A$ denote the area under the curve from the fixed boundary $x = 1$ to the variable boundary $x = x$.

Then $$\frac{dA}{dx} = y = x^2.$$

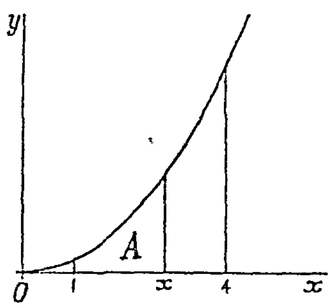



FIG. 4.3

Therefore $A = \frac{1}{3}x^3+c$, where $c$ is a constant.

To find $c$ we must know a pair of values of $A$ and $x$. If the variable boundary is made to coincide with the fixed one $x = 1$, the area between them is 0. Hence when $x = 1$, $A = 0$.

Therefore $c$ is given by $0 = \frac{1}{3}+c$, i.e. $c = -\frac{1}{3}$.

Hence $A = \frac{1}{3}x^3-\frac{1}{3}$.

The required area from $x = 1$ to $x = 4$ is found by making the variable boundary coincide with $x = 4$, i.e. by putting $x = 4$ in the formula for $A$.

$$\text{Required area} = \frac{4^3}{3}-\frac{1}{3} = 21.$$

As in section 1.20 the answer is usually left as a number and the unit is not stated.

If, however, Fig 4 3 is the plan of a field on a scale of 1 chain = unit $x$ and unit $y$, the area may easily be found in any of the usual units of area In the inequality

$$y\Delta x < \Delta A < (y+\Delta y)\Delta x$$

it is implied that $\Delta A$, and therefore $A$, is measured in the same unit as the rectangles $y\Delta x$ and $(y+\Delta y)\Delta x$ But these areas are measured in terms of a unit rectangle whose dimensions parallel to the $x$ and $y$ axes are each 1 (We say rectangle, and not square, to cover the case when the scales employed for $x$ and $y$ are not the same ) Hence the area of the field is 21 unit rectangles = 21 sq ch = 2 1 acres

EXERCISE 4 A

1–9 Find the areas under the following curves

1 $y = x^2$ from $x = 2$ to $x = 5$

2 $y = 4x^3$ from $x = 1$ to $x = 3$

3 $y = 6x^2+2x$ from $x = 1$ to $x = 1\frac{1}{2}$

4 $y = x^4+1$ from $x = 1$ to $x = 2$

5 $y = x^2+4x+3$ from $x = -1$ to $x = 1$

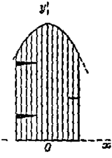
6 $y = 20-3x^2$ from $x = -2$ to $x = 1$

7 $y = 1+6x^2$ from $x = 0$ to $x = 2$

8 $y = 8-2x-x^2$ from $x = -2$ to $x = 0$

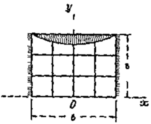
9 $y = x^4+5$ from $x = -2$ to $x = 2$

10 A field is bounded by three straight hedges, $x = 0$, $y = 0$, $x = 10$, and a curved road, $y = 1\ 2+1\ 8x-0\ 15x^2$, the unit of $x$ and $y$ being 1 chain Find the area of the field in acres



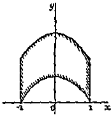

Ex 4 A 11

11 A church door is bounded by part of the curve $y = 11-x^2/3$ and the lines $x = 3$, $x = -3$, $y = 0$, where $x$, $y$ are measured in feet and the axes are placed as in the sketch Find the area of the door




Ex. 4 A 12

12 A pelmet is made to cover the curtain rings above a window, its boundary being $y = 5+x^2/16$ ($x$, $y$ in feet and the axes as in the sketch) Find the area of the part of the window through which light can still pass What is the area of the pelmet ?




Ex 4 A 13

13 Find the area bounded by $y = 12-3x^2$, $x = 2$, $y = 32-3x^2$, $x+2 = 0$

4.3. Example. Find the area between the curve $y = 6(1-x^2)$ and the $x$-axis.

*Solution.* Sketch the curve.

It is now clear that we have to find the area under the curve from $x = -1$ to $x = 1$.

Let the area under the curve from $x = -1$ to $x = x$ $(\leqslant 1)$ be $A$.

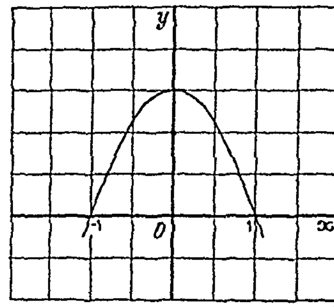



Fig. 4.4

Then $$\frac{dA}{dx} = 6(1-x^2),$$

$$A = 6x-2x^3+c,$$

where $c$ is a constant.

But $A = 0$ when $x = -1$.

Therefore $$0 = -6+2+c$$

and $$c = 4.$$

Therefore $$A = 6x-2x^3+4.$$

The required area is the value of $A$ when $x = 1$. The area under the curve from $x = -1$ to $x = 1$ is $6-2+4 = 8$.

## Exercise 4.B

1. Find the area between the curve $y = 4-x^2$ and the $x$-axis.
2. Find the area between $y = 20(1-x^4)$ and the $x$-axis.
3. Find where the curve $y = 4x-3-x^2$ crosses the $x$-axis and find the value of $x$ where $y$ is a maximum. Draw a sketch.

   Show that a line parallel to the $y$-axis through the maximum point divides the area between the curve and the $x$-axis into two equal parts.
4. Find the area in the first quadrant bounded by $y = 9-x^2$ and the axes.
5. Find the points of intersection of the curve $y = 7+3x-3x^2$ and the straight line $y = 1$. Find the area bounded by the curve and the straight line.
6. Find the area bounded by the curve $y+3x^2 = 48$ and lying in the first quadrant.

4.4. The result, $dA/dx = y$, proved in 4.2 is very important. In this section we look at it again from a practical point of view.

A machine moves along a straight road spraying with tar a 10-ft. width of the road as it goes. If the area sprayed, after the machine has travelled $x$ ft., is $a$ sq. ft. then

$$a = 10x,$$

and $$\frac{da}{dx} = 10.$$

Three boundaries of the area $a$ are fixed and the fourth moves The rate of increase of $a$ with $x$ equals the width of the moving boundary

Compare this with the result of 4 2 There, the area $A$ also had three fixed boundaries If we think of the variable boundary as moving in the direction in which $x$ increases, we can state the equation $dA/dx = y$ in the form the rate of increase of $A$ with $x$ is given by the length of the moving boundary Since, in general, $y$ changes as $x$ increases, the rate of increase of $A$ with $x$ is variable

A further point, requiring care, is illustrated by the following attempt to find the area under the curve $y = x(x-1)(x-2)$ from $x = 0$ to $x = 2$

If $A$ represents the area from $x = 0$ to $x = x$ and we use the formula $dA/dx = y$, we have

$$\frac{dA}{dx} = x^3-3x^2+2x,$$

$$A = \frac{x^4}{4}-x^3+x^2+c$$

Since $A = 0$ when $x = 0$, $c = 0$

Therefore putting $x = 2$, the required area is

$$\frac{2^4}{4}-2^3+2^2 = 4-8+4 = 0$$

To find the reason for this absurd result we sketch the curve

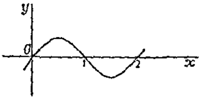



FIG 4 5

From $x = 0$ to $x = 1$ $y$ is positive, while from $x = 1$ to $x = 2$ $y$ is negative Thus the equation $dA/dx = y$ gives a positive value to $dA/dx$ when $x$ is between 0 and 1 and a negative value when $x$ is between 1 and 2 This means that $A$ increases as the variable boundary $x = x$ moves from $x = 0$ to $x = 1$ but decreases as it moves from $x = 1$ to $x = 2$ The zero result obtained above shows that the decrease from $x = 1$ to $x = 2$ exactly cancels the increase from $x = 0$ to $x = 1$

Fig 4 5 shows that the example was incorrectly worded It is not clear what is meant by the area *under* the curve from $x = 0$ to $x = 2$ If the meaning is the area between the curve and the $x$ axis, this area cannot be found by applying the formula $dA/dx = y$ in the direct manner which we tried at the beginning of this example The evaluation of the area in cases when $y$

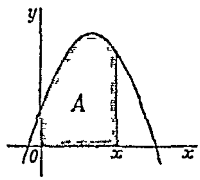
takes negative values will be considered later. The purpose of the present example is to emphasize the importance of drawing a sketch when an area is calculated.

## Exercise 4.C

1. The area $A$ is bounded by $y = 0$, $x = 0$, $y = 1+x^2/4$ and the variable line $x = x$. What is the rate at which $A$ increases with $x$ when $x = 2$ ?

2. The area $A$ is bounded by $y = 0$, $x = -1$, $y = x^3+4$, and the variable line $x = u$. Find the rates at which $A$ increases with $x$ (i) when $u = 0$, (ii) when $u = 1$, (iii) when $u = 10$.

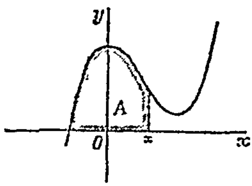
3. The area, $A$, is bounded by the axes, the curve $y = 4+6x-x^2$, and a variable boundary parallel to the $y$-axis and moving in the direction of increasing $x$. Find the value of $x$ when the rate of increase of the area is a maximum.




4. The area, $A$, is bounded by

$$y = 4x^3-12x^2+20,\ y = 0,\ x = -1,$$

and a variable boundary parallel to the $y$-axis. Find $x$ when the rate of increase of $A$ with $x$ is (i) a maximum, (ii) a minimum.



## V

## SUMMATION

### 5 1. The area under a curve as the limit of a sum of rectangular areas

In section 4 1 we said that the area of a rectangle is the fundamental area on which we seek to base the calculation of the area of other geometrical figures In this section we consider, again, the meaning of the area under a curve in terms of the areas of rectangles adopting a different point of view which leads to more far reaching applications of this kind of calculation

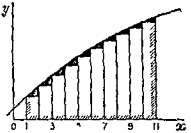



Fig 5 1 Lower sum with 10 rectangles

Consider the area under a curve from $x = 1$ to $x = 11$ and suppose, to begin with that the curve rises steadily and $y$ is positive in this range of $x$

Divide the area into 10 strips of equal width by drawing parallels to the $y$ axis through $x = 2$ 3 , 10 Draw parallels to the $x$ axis through the top of the left hand upright of each strip forming 10 rectangles under the curve Then if we add together the areas of these rectangles we have a rough approximation to the area under the curve We call a sum of rectangular areas formed in this way a sum of *lower* rectangles or, briefly, a *lower sum* Clearly the lower sum with 10 rectangles already formed is less than the area under the curve by the sum of the 10 'almost triangular' areas shaded in Fig 5 1 We cannot calculate these areas because our fundamental difficulty remains, each area has a curved boundary

Now suppose that we form a new set of rectangles by drawing a parallel to the $x$ axis through the top of the right hand upright of each strip

Adding the areas of these rectangles we obtain another rough approximation which is certainly greater than the area under the curve We call a set of rectangular areas of this kind a sum of *upper* rectangles or an *upper sum*

We can now say that the area under the curve lies between the upper and lower sums with 10 rectangles. The difference between these sums is the sum of the areas of the 10 shaded rectangles through which the curve passes in Fig. 5.3.

As these shaded rectangles are all of the same width we can imagine them piled on top of the first to form one rectangle of width 1. The top of the pile will be level with the point on the curve at $x = 11$. Hence, if $y = h$ when $x = 1$ and $y = k$ when $x = 11$, the sum of the 10 small rectangles is $(k-h)\times 1 = k-h$. Now the area under the curve is between the upper and lower sums which differ by $k-h$. Hence either the upper or the lower

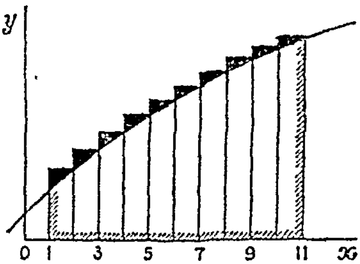



FIG. 5.2. Upper sum, with 10 rectangles

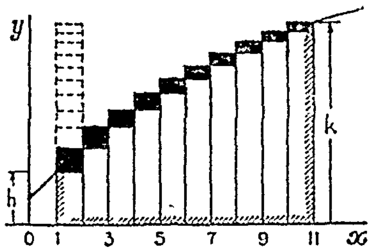



FIG. 5.3

sum with 10 rectangles is an approximation to the area under the curve with an error which is certainly less than $k-h$.

It has already been pointed out that we cannot calculate directly the difference between either of these approximations and the area. Therefore our only hope of calculating the area more precisely lies in finding two closer approximations. The method of doing this which seems most likely to be successful consists in increasing the number of rectangles in each sum. We therefore divide the area under the curve from $x = 1$ to $x = 11$ into 100 strips of equal width and form an upper and a lower sum each containing 100 rectangles. It is impracticable to represent these rectangles on a figure but, if we refer to Fig. 5.3 and imagine the number of strips increased to 100, we can see that the difference between the new lower and upper sums is the sum of 100 small rectangles which, piled on top of the first, would still stretch from $y = h$ to $y = k$. But the width of each rectangle is now $10/100 = 1/10$, so that the difference between the upper and lower sums is now $(k-h)/10$. Therefore each sum differs from the area under the curve by less than $(k-h)/10$, which is only $\frac{1}{10}$ of the corresponding difference when the number of rectangles is 10.

Let the number of rectangles be increased to 1000 so that the width of each is decreased to $10/1000 = 1/100$ The difference between the upper and lower sums with 1000 rectangles is $(k-h)/100$ Thus as the number of strips increases, the difference between the upper and lower sums decreases, i e they move towards each other but in such a way that the area under the curve remains between them Also we can make their difference as small as we like by taking a sufficiently large number of rectangles For, if we consider $(k-h)/100$ to be too large a difference, we have only to take 10 times as many rectangles to obtain a difference 10 times as small, i e $(k-h)/1000$ If we still think that this is too large, we again multiply the number of rectangles by 10 and the difference between the upper and lower sums becomes $(k-h)/10000$ This argument may be repeated as many times as we like

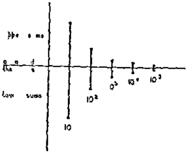


Fig 5 4

The situation may now be summed up in Fig 5 4, which is not drawn to scale

In this figure each pair of points represents, on a vertical scale, the values of the upper and lower sums, formed for a number of rectangles indicated by the number below the pair of points The horizontal line represents, on the same vertical scale, the area under the curve

It is clear that the upper and lower sums close down on one another as the number of rectangles is increased, and by taking a sufficiently large number of rectangles we can find a pair of corresponding points which are separated by as small a vertical distance as we like But the area under the curve is between each pair of sums Hence, in Fig 5 4, one and only one horizontal line can be drawn which is between every pair of points *Therefore the lower sums increase up to the area and the upper sums decrease down to the area as the number of rectangles is increased beyond any number, however large* When we say that the number of rectangles is increased beyond any number, however large, we mean that it is increased so as to exceed any number that can be named, i e beyond 1 million, 10 million, 1 million million, etc

* For those who wish to consider the passage to the limit more carefully, we add the following, more formal, statement

Fig. 5.4 is intended to convey, graphically, the meaning of the facts which have emerged in the preceding discussion. These are

(1) *There is a number, the area under the curve, which is less than each upper sum and greater than each lower sum.*

Thus each pair of points in Fig. 5.4 'straddles' the horizontal line, and a point is above or below the line according as it represents an upper or lower sum.

(2) *The difference between an upper sum and the corresponding lower sum decreases as the number of rectangles increases and can be made as small as we like by taking a sufficiently large number of rectangles.*

Thus each pair of points (after the first) are nearer together than the preceding pair. (Actually, the distance between a pair of points is $\frac{1}{10}$ of the distance between the preceding pair, but this cannot conveniently be shown on the diagram.)

Also if any small distance, $e$, is chosen on the vertical scale, all the pairs after a certain one are nearer together than $e$.

If we now fix our attention on the points representing the lower sums, it follows from (2) that we can find a point as near as we like to the point representing the corresponding upper sum and that this is then true of all subsequent pairs. Therefore from (1) these points are still nearer the horizontal line, and, since they represent lower sums, cannot cross it.

*Therefore the lower sums tend to the area under the curve as a limit, when the number of rectangles is increased beyond any number, however large.* A similar argument may be used to show that the limit of the upper sums is, also, the area under the curve.

*Notes*

1. It is important that it should be fully understood that the lower sums increase up to the area under the curve in their own right, so to speak, and similarly that the upper sums decrease down to the area as the number of rectangles is increased. It may be useful to show this for the lower sums by an argument in which the appeal to the upper sums is delayed as long as possible. A similar argument may easily be constructed for the upper sums.

When we construct the lower sum with 100 rectangles, exactly 10 of these rectangles fall in each of the 10 strips into which the area under the curve was previously divided to form the lower sum with 10 rectangles. Fig. 5.5 shows, on a magnified scale,

the strip from $x = 3$ to $x = 4$ divided at $x = 3\ 1, 3\ 2, \ , 3\ 9$, into 10 smaller strips

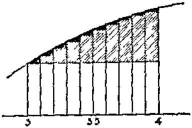



Fig 5 5

It is clear that the sum of the 10 lower rectangles falling in this strip is greater than the area of the one lower rectangle belonging to the lower sum with 10 rectangles By applying this argument to each of the 10 strips, $x = 1$ to $x = 2, \ , x = 10$ to $x = 11$, we see that the lower sum with 100 rectangles is greater than the lower sum with 10 rectangles Similarly the lower sum with 1000 rectangles is greater than the lower sum with 100 rectangles This argument may be repeated as many times as we like

Also it is clear that the lower sums with 10, 100, 1000 rectangles are each less than the area under the curve

Now suppose we represent the values of the lower sums and the area under the curve upon a straight line with a suitable scale Let $O$ be taken as origin and let $A$ be a point such that $OA$ represents the area under the curve Let $s_{10}$ be a point such that $Os_{10}$ represents the value of the lower sum with 10 rectangles

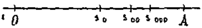



Fig 5 6

Then the point $s_{100}$ representing the lower sum with 100 rectangles, lies to the right of $s_{10}$ (it represents a greater sum) and to the left of $A$ Similarly $s_{1000}$ lies to the right of $s_{100}$ and to the left of $A$ Suppose we go on to plot the points $s_{10000}, s_{100000},$ Each of these points lies between those previously constructed and $A$ It is clear that the points crowd together in the neighbourhood of a point which is *either A or to the left of A*

This is where we have to appeal to the upper sums For, when we have shown that upper and lower sums can be found with as small a difference as we like, it follows, since $A$ is between each pair of sums, that lower sums can be found which differ from $A$ by as small a number as we like Hence the limit of the points in Fig 5 6 is $A$ itself and not a point to the left of $A$

2 As a further illustration of the meaning of a limit we may consider the problem of drawing Fig 5 4 to scale Suppose the vertical scale is chosen so that the first pair of points are 1 in apart Then the second pair are $\frac{1}{10}$ in apart, the third pair are

$\frac{1}{100}$ in. apart, etc. Now $\frac{1}{100}$ in. is about the limit at which the human eye can separate two fine marks. Hence we can safely say that, on this scale, Fig. 5.4 would have 3 pairs of points which could be distinguished and that the remainder would be indistinguishable from each other and, therefore, from the horizontal straight line passing between them. The effect of a change of scale is, merely, to alter the number of points which can be distinguished. To take an extreme example, suppose the first pair of points are 1 mile apart. Since 1 mile = 63360 in., the seventh pair of points would then be 0·06 in. apart; the eighth pair would be 0·006 in. ($< 0{\cdot}01$) apart. We could then distinguish 7 pairs of points and all the rest (too numerous to count) would merge into the horizontal straight line.

### Exercise 5.A

Find the area under $y = 4x^2$ from $x = 1$ to $x = 3$ by the method of Chapter IV.

Calculate the upper and lower sums for this area when the number of rectangles is (i) 2, (ii) 4, and verify that the area under the curve is between each pair of sums. Note that (*a*) the upper sum for 4 rectangles is less than the upper sum for 2 rectangles; (*b*) the lower sum for 4 rectangles is greater than the lower sum for 2 rectangles; (*c*) as the number of rectangles increases from 2 to 4 the upper and lower sums approach one another.

**5.2.** The work of section 5.1 was based on a rising curve. If the curve falls throughout the range of $x$ considered, the argument holds with slight modifications (e.g. right-hand upright for left-hand upright, etc.). If the curve contains maximum and minimum values within the range of $x$, it must be broken up so that on each stretch the curve rises or falls.† The argument is then applied to each stretch.

## 5.3. The Integral

We have seen that the area under a curve from $x = a$ to $x = b$ is the limit of a sum of rectangles stretching from $x = a$ to $x = b$ as the number of rectangles becomes increasingly large. It is not practicable to draw all the rectangles when they are numerous. We therefore draw one, to show the type of rectangle

† This is possible except when the curve has an infinite number of maxima and minima in a finite range of $x$, e.g. $y = \sin (36/x)°$ between $x = 0$ and $x = 1$. (See Section 2.13, Fig. 2.20.)

that we are considering (upper or lower), and imagine the others This rectangle may be called the *typical rectangle*

In Fig 5 7, $PQMN$ is a typical lower rectangle $P$ is any point on the curve between $x = a$ and $x = b$ Let this be the point $(x, y)$ so that $ON = x$† and $PN = y$ The width of the rectangle,

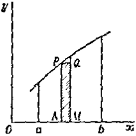


Fig 5 7

$$NM = \frac{b-a}{\text{number of rectangles}}.$$

But a more convenient expression is found by regarding $NM$ as an increment of $ON = x$ and writing

$$NM = \delta x \quad \text{or} \quad dx.$$

Then the area of the rectangle

$$PQMN = y\,dx$$

The area of the curve from $x = a$ to $x = b$ is the limit of the sum of areas like $y\,dx$, the summation extending from $x = a$ to $x = b$ and the limit being taken as the number of rectangles increases beyond any number, however large, so that $dx \to 0$

This limit is called an *integral* and it is denoted by the symbol

$$\int_a^b y\,dx$$

This is read 'integral from $a$ to $b$ of $y\,dx$' The *integral sign*, $\int$, is the old form of the letter $s$ and was first used by the German mathematician Leibnitz, about 1675 He chose $s$ as the initial letter of *summa* (scientific works were generally written in Latin at that time) to remind us that it represents the limit of a *sum* of rectangles, of which a typical one is described by the rest of the expression The process of evaluating the integral is called *integration* The numbers at the top and bottom of the integral sign are called the *limits of integration* and the integration is said to be performed from $a$ to $b$

Example Evaluate the integral, $\int_1^4 x^2\,dx$

*Solution* Putting $y = x^2$, the integral becomes $\int_1^4 y\,dx$ It is, therefore, the limit of the sum of rectangular areas like $y\,dx$, the summation extending from $x = 1$ to $x = 4$ This is the area under the curve $y = x^2$ from $x = 1$ to $x = 4$

† Strictly, $x$ can only have values commensurate with $(b-a)$ This difficulty is ultimately overcome by the use of a more general argument in which the rectangles are not all of equal width

Denote the area under the curve from $x = 1$ to $x = x$ by $A$. Then

$$\frac{dA}{dx} = y = x^2.$$

Therefore $$A = \frac{x^3}{3}+c,$$

$$A = 0 \text{ when } x = 1.$$

Therefore $0 = \frac{1}{3}+c,\ c = -\frac{1}{3}$,

$$A = \frac{x^3}{3}-\frac{1}{3}.$$

Putting $x = 4$, we obtain the required area.

Hence $$\int\limits_1^4 x^2\,dx = \frac{4^3}{3}-\frac{1}{3} = 21.$$

Fig. 5.8

### Exercise 5.B

1. What area is represented by $\int\limits_1^2 x^2\,dx$? Find the value of this integral.

2–4. Evaluate the following integrals:

2. $\int\limits_2^5 x^2\,dx.$ 3. $\int\limits_1^3 4x^3\,dx.$ 4. $\int\limits_1^{11} (1+2x)\,dx.$

## 5.4. The value of $\int\limits_a^b x^2\,dx$

Putting $y = x^2$, $\int\limits_a^b x^2\,dx = \int\limits_a^b y\,dx$ and is seen to be the limit of the sum of rectangular areas like $y\,dx$ from $x = a$ to $x = b$. This limit is the area under the curve $y = x^2$ from $x = a$ to $x = b$. Let the area under the curve from $x = a$ to the variable boundary $x = x$ be $A$. Then

$$\frac{dA}{dx} = y = x^2,$$

$$A = \frac{x^3}{3}+c.$$

Since $A = 0$ when $x = a$,

$$0 = \frac{a^3}{3}+c \quad \text{or} \quad c = -\frac{a^3}{3}.$$

Therefore, $$A = \frac{x^3}{3}-\frac{a^3}{3},$$

and the required area is $\frac{1}{3}b^3-\frac{1}{3}a^3$.

Therefore $$\int_a^b x^2\,dx = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3$$

We can now see that the evaluation of $\int_a^b x^2\,dx$ consists of four steps

(1) Find the function which has $x^2$ for its gradient function
(2) Find the value of this function when $x = b$
(3) Find the value of this function when $x = a$
(4) Subtract (3) from (2)

The work may be set out, concisely, like this

$$\int_a^b x^2\,dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

*Note* $\tfrac{1}{3}x^3$ is not the only function which has gradient function $x^2$ The most general function would be $\tfrac{1}{3}x^3 + k$, where $k$ is a constant The use of this general function, however, leads to the same result

$$\int_a^b x^2\,dx = \left[\frac{x^3}{3} + k\right]_a^b = \frac{b^3}{3} + k - \frac{a^3}{3} - k = \frac{b^3}{3} - \frac{a^3}{3}$$

The evaluation of $\int_a^b x^2\,dx$ explained in this section is an example of the general method for evaluating integrals This consists of the same four steps except that a different function is substituted for $x^2$

EXAMPLE Evaluate $\int_1^2 24x^3\,dx$

*Solution*

$$\begin{aligned}\int_1^2 24x^3\,dx &= [6x^4]_1^2\\ &= 6\;2^4 - 6\;1^4\\ &= 96 - 6\\ &= 90\end{aligned}$$

EXERCISE 5 C

1–12 Evaluate the integrals

1 $\int_2^4 3x^2\,dx$ 2 $\int_1^7 x\,dx$ 3 $\int_1^4 (1+2x^2)\,dx$

4 $\int_1^5 (1/x^\circ)\,dx$ 5 $\int_{-1}^2 3x^4\,dx$ 6 $\int_0^1 (x^3+x)\,dx$

7. $\int_{-5}^{-2} 6x^2\,dx$.

8. $\int_{1}^{2} (3x^2-x)\,dx$.

9. $\int_{5}^{10} (1-10/x^3)\,dx$.

10. $\int_{1}^{4} (4x+1/x^2)\,dx$.

11. $\int_{-2}^{3} (x+2)(3-x)\,dx$.

12. $\int_{-2}^{-\frac{1}{2}} (x^2+1/x^2)\,dx$.

13. Evaluate
$$\int_{\frac{1}{2}}^{4} (2x+1/x^2)\,dx.$$

*Write down* the value of $\int_{\frac{1}{2}}^{4} \frac{2x^3+1}{x^2}\,dx$.

14. Evaluate

(a) $\int_{1}^{2} \frac{9x^4+1}{x^2}\,dx$; (b) $\int_{2}^{3} \frac{9x^4+1}{x^2}\,dx$; (c) $\int_{1}^{3} \frac{9x^4+1}{x^2}\,dx$.

Verify that
$$\int_{1}^{2} \frac{9x^4+1}{x^2}\,dx+\int_{2}^{3} \frac{9x^4+1}{x^2}\,dx = \int_{1}^{3} \frac{9x^4+1}{x^2}\,dx.$$

15. Verify the following statements by working out the integrals:

(a) $\int_{2}^{4} 2x\,dx = 2\int_{2}^{4} x\,dx$.

(b) $\int_{-2}^{-1} (3x^2-3)\,dx = 3\int_{-2}^{-1} (x^2-1)\,dx$.

(c) $\int_{1}^{3} \frac{5(x+1)}{4}\,dx = \frac{5}{4}\int_{1}^{3} (x+1)\,dx$.

(d) $\int_{1}^{2} \pi x^2\,dx = \pi\int_{1}^{2} x^2\,dx$.

16. By finding the value of each integral, show that
$$\int_{1}^{2} \left(x+\frac{12}{x^4}\right)dx = \int_{\frac{1}{2}}^{1} \left(12x^2+\frac{1}{x^3}\right)dx.$$

## 5 5. The calculation of an area by means of an integral

The use of an integral enables us to shorten very considerably the calculation of the area under a curve

Example 1 Find the area under the curve $y = x^3+6x+8$ from $x = -1$ to $x = 2$

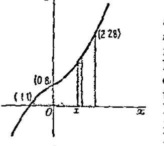



Fig 5 9

*Solution* Fig 5 9 is a sketch of the curve $y = x^3+6x+8$ showing the area which it is required to find

We imagine that a set of lower rectangles is constructed from $x = -1$ to $x = 2$, showing on our figure the typical rectangle whose area is $y\,dx$ Hence the lower sum is the sum of areas like $y\,dx$ extending from $x = -1$ to $x = 2$ But the required area is the limit of this lower sum as the number of rectangles increases beyond any number, however large, and $dx \to 0$ By the definition of the integral, this limit is $\int_{-1}^{2} y\,dx$

Therefore the area under the curve from $x = -1$ to $x = 2$ is

$$\begin{aligned}\int_{-1}^{2} y\,dx &= \int_{-1}^{2} (x^3+6x+8)\,dx \\ &= \left[\frac{x^4}{4}+3x^2+8x\right]_{-1}^{2} \\ &= 4+12+16-(\tfrac{1}{4}+3-8) \\ &= 36\tfrac{3}{4}\end{aligned}$$

Example 2 Find the area in the first quadrant bounded by $y = 9-x^2$ and the coordinate axes

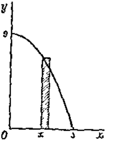



Fig 5 10

*Solution* The curve cuts the $x$ axis where $x = 3$ and the $y$ axis where $y = 9$ We require the area under the curve from $x = 0$ to $x = 3$

We know that the upper and lower sums both tend to the area under the curve as a limit and hence we choose the type of sum which can be expressed most simply In this case, the area of the typical upper rectangle is $y\,dx$, and the area of the typical lower rectangle is $(y+\delta y)\,dx$ We therefore choose the upper sum

The area under the curve from $x = 0$ to $x = 3$ is the *limit of the upper sum, formed* from $x = 0$ to $x = 3$, as the number of rectangles increases beyond any number, however large, and $dx \to 0$

Hence the area is $$\int_0^3 y\,dx$$

which is $$\int_0^3 (9-x^2)\,dx = \left[9x-\frac{x^3}{3}\right]_0^3 = 27-9 = 18.$$

## Exercise 5.D

1. Find the area under the curve $y = 1+x^4$ from $x = -1$ to $x = +1$. Verify that it is twice the area from $x = 0$ to $x = 1$. Of what property of the curve is this a consequence?

2. Find the area under the curve $y = 2x+3x^2$ from $x = \frac{1}{2}$ to $x = 1\frac{1}{2}$.

3. Find the area under the curve $y = 2x+16x^3$ from $x = \frac{1}{2}$ to $x = 2$.

4. Find the area under the curve $y = 4-x^2-x^3$ from $x = 0$ to $x = 1$.

5. Find the areas under the curves $y = 1/x^2$, $y = 1/x^4$ from $x = 1$ to $x = 10$. The curves do not intersect when $x > 1$; which one keeps closer to the $x$-axis.

6. Find the area under the curve $y = x-2/x^3$ from $x = 2$ to $x = 3$.

7. Find the area between the curve $y = 9+6x-3x^2$ and the $x$-axis.

8. Find the area between the curve $y = (1-x)(x+2)$ and the $x$-axis.

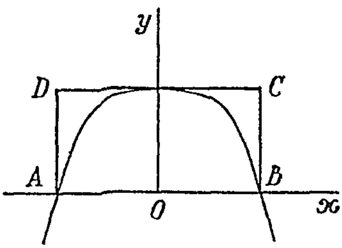
9. Find the area under the curve $y = 1+1/x^2+1/x^3$ from $x = \frac{1}{2}$ to $x = 1$.

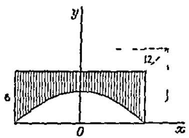
10. Find the area under the curve $y = 4-x^2$ from $x = -1$ to $x = 0$.

11. Find the area under the curve $y = x^3+3x^2+1$ from $x = -3$ to $x = 1$.

12. Show that the curve $y = x^2$ divides the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$ into two parts whose areas are in the ratio 2:1.

13. The curve $y = 1-x^4$ cuts the $x$-axis at $A$ and $B$. The rectangle $ABCD$ is completed, $CD$ being the tangent to the curve at (0, 1). Find the ratio of the areas of the two parts into which the rectangle is divided by the curve.






14 The arch of a stone bridge is a portion of the curve

$$20y+x^2-100=0$$

with the axes in the position shown in the sketch Assuming that the portion of the bridge shown by shading is solid 12 ft wide, and made of stone weighing 164 lb per cu ft, find its weight in tons

15 Find the maximum and minimum values of $y=4x^3+3x^2+1$ and sketch the curve Find the area under the curve from $x=-1$ to $x=2$

## 5 6 Stationary values within the range of integration may be ignored

We noted in section 5 2 that the argument of section 5 1 holds if the curve rises or falls steadily between the limits of integration If there are stationary values between these limits we ought to find the area under each rising or falling portion of the curve and add the results to find the required area

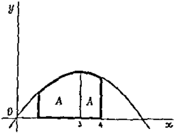



FIG 5 11

This point arises several times in Ex 5 D You probably ignored it and yet obtained the correct answer The reason for this must now be considered

Find the area under the curve $y=6x-x^2$ from $x=1$ to $x=4$

It is easily seen that $y$ has a maximum value at $x=3$ Denote the areas under the curve from $x=1$ to $x=3$ and from $x=3$ to $x=4$ by $A_1$ and $A_2$ Then the required area is $A_1+A_2$

$$A_1=\int_1^3(6x-x^2)\,dx$$

$$=\left[3x^2-\frac{x^3}{3}\right]_1^3=3\cdot 3^2-\frac{3^3}{3}-(3-\tfrac{1}{3})$$

$$A_2=\int_3^4(6x-x^2)\,dx$$

$$=\left[3x^2-\frac{x^3}{3}\right]_3^4=3\cdot 4^2-\frac{4^3}{3}-\left(3\cdot 3^2-\frac{3^3}{3}\right)$$

When we add, the expression $3.3^2-\frac{3^3}{3}$ cancels, and we have

$$A_1+A_2 = 3.4^2-\frac{4^3}{3}-(3-\tfrac{1}{3}).$$

$$\text{But } \int_1^4 (6x-x^2)\,dx = \left[3x^2-\frac{x^3}{3}\right]_1^4 = 3.4^2-\frac{4^3}{3}-(3-\tfrac{1}{3}).$$

Hence, in practice, even if stationary values occur between the limits of integration, we may integrate in the ordinary way and obtain the correct value. The setting out of the example above is intended to make it clear that the cancelling, which occurs there, will always occur whatever expression is being integrated.

**5.7.** We now consider the evaluation of areas not situated so conveniently with respect to the axes.

EXAMPLE. Find the area between the curve $4y = x^2$ and the straight line $y = x$.

*Solution.* The curve and line meet where $4x = x^2$, i.e. where $x = 0$ and $x = 4$. Hence the intersections are the origin and (4, 4). Let these points be $O$ and $A$ and draw $AB$ perpendicular to the $x$-axis.

The required area

= area of triangle $OAB$—area under the curve $4y = x^2$ from $x = 0$ to $x = 4$

$$= 8-\int_0^4 \frac{x^2}{4}\,dx = 8-\left[\frac{x^3}{12}\right]_0^4$$

$$= 8-5\tfrac{1}{3} = 2\tfrac{2}{3}.$$

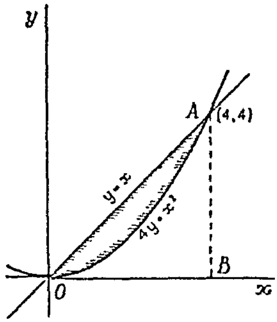



FIG. 5.12

Now let $OR$ represent any value of $x$ between 0 and 4. Then the area under the curve is the limit of the sum of rectangles like $LMSR$ stretching from $x = 0$ to $x = 4$. Also the area under the straight line is the limit of the sum of rectangles like $PQSR$ stretching from $x = 0$ to $x = 4$. Therefore the required area is the limit of the sum of rectangles like $PQML$ stretching from $x = 0$ to $x = 4$ as the number of rectangles increases beyond any number, however large.

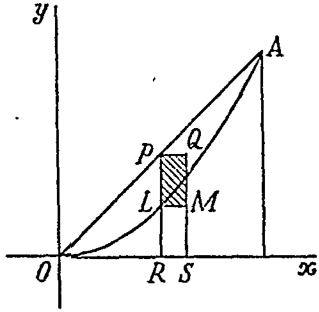



FIG. 5.13

Therefore the required area is

$$\int_0^4 \left(x-\frac{x^2}{4}\right) dx = \left[\frac{x^2}{2}-\frac{x^3}{12}\right]_0^4 = 8-5\tfrac{1}{3} = 2\tfrac{2}{3}.$$

Thus the method of obtaining the integral required to effect the summation, by first writing down the typical element of the sum, applies even when we approximate at both the top and bottom of the rectangle

### Exercise 5 E

1 Find the area in the first quadrant between $y = 4x$ and $y = x^3$

2 Find the area bounded by $y = x(10-x)$ and $y = 4x$

3 Find the area bounded by $y = 2x^2$, $x = 0$, $y = 8$

4 Find the area bounded by $y = 10x^4$ and $y = 10$

5 Find the area in the first quadrant between $2y = x^3$ and $y = 2x^2$

6 Find the area in the first quadrant between $3y = x^2$ and $y = 3x$

7 Find the area between the curve $y = x^2-2x+2$ and the chord $y+x = 4$

8 Find the area bounded by $y = x^2$, $y = 1/x^2$, and $x = 3$

9 Find the area between $y = 4x^2$ and $y = 5-x^2$

10 The stiffening material inside the peak of a schoolboy's cap has the shape bounded by the curves $4y = x^2$, $8y = x^2+16$, where unit $x$ and $y = 1$ in Find its area

## 5 8. The use of integrals

EXAMPLE A contractor is required to dig a well of cross section 1 sq yd and depth 10 yd, and he knows that the cost of excavating soil from a depth of $x$ yd is $(1+x^2)$ shillings per cu yd What is the cost of digging the well?

*First solution* Let the total cost of digging to a depth of $x$ yd be $c$ shillings Then $c$ depends on $x$ so that the cost of digging down $(x+\delta x)$ yd may be represented by $(c+\delta c)$ shillings and the cost of digging out the layer of soil between the depths $x$ yd and $(x+\delta x)$ yd is $\delta c$ shillings The volume of this soil is $\delta x \times 1 = \delta x$ cu yd and the cost of excavating it varies from $(1+x^2)$ shillings per cu yd at depth $x$ yd to $[1+(x+\delta x)^2]$ shillings per cu yd at depth $(x+\delta x)$ yd The cheapest rate applicable to the layer is $(1+x^2)$ shillings per cu yd, so that if we apply this rate to the whole layer we obtain an estimate of the cost which is too low Similarly, by using the dearest rate, $[1+(x+\delta x)^2]$ shillings per cu yd, we obtain an estimate which is too high

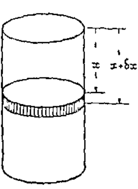


FIG 5 14

Hence $(1+x^2)\,\delta x < \delta c < [1+(x+\delta x)^2]\,\delta x$

Dividing by $\delta x$

$$(1+x^2) < \frac{\delta c}{\delta x} < 1+(x+\delta x)^2,$$

and, now, letting $\delta x \to 0$, we have

$$\frac{dc}{dx} = 1+x^2.$$

Therefore, $$c = x+\frac{x^3}{3}+k,$$

where $k$ is a constant.

But $c = 0$ when $x = 0$. Therefore $k = 0$, and $c = x+\frac{x^3}{3}$.

The cost of digging the well is found by putting $x = 10$ and is, therefore,

$$10+\frac{10^3}{3} = 343\tfrac{1}{3} \text{ shillings}$$
$$= \pounds 17.\ 3s.\ 4d.$$

*Second solution.* Now, represent the cost per cu. yd. by $y$ shillings, so that $y = 1+x^2$ and draw the graph of this equation.

Then, when we find the approximations $(1+x^2)\,\delta x$ and $[1+(x+\delta x)^2]\,\delta x$ to the cost of digging out a layer by holding the rate per cu. yd. constant at its cheapest and dearest values for the layer, we are merely calculating typical lower and upper rectangles for the curve $y = 1+x^2$. If we form the upper and lower sums of rectangles from $x = 0$ to $x = 10$ we know

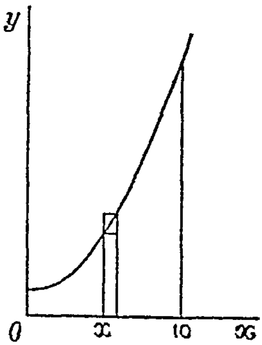



FIG. 5.15

(a) that the total cost of digging the well is between these sums;

(b) that the limits of these sums, as their number increases beyond any number, however large, and $\delta x \to 0$, have the same value, the area under the curve from $x = 0$ to $x = 10$, and that this value is represented by $\int_0^{10} (1+x^2)\,dx$. (It is, of course, assumed that there is a definite area under the curve.)

Therefore the cost of digging out the well

$$= \int_0^{10} (1+x^2)\,dx = \left[x+\frac{x^3}{3}\right]_0^{10} = 343\tfrac{1}{3} \text{ shillings.}$$

In the second solution the emphasis is upon summation. Consider the following simplified form of this solution.

*Intuitive solution.* The cost of excavating the layer between the depths $x$ yd. and $(x+\delta x)$ yd. is *approximately* $(1+x^2)\,\delta x$ shillings. We require the limit of the sum of the costs of all such

layers from $x = 0$ to $x = 10$ as the number of layers increases beyond any number, however large, and $\delta x \to 0$ This limit is

$$\int_0^{10} (1+x^2)\,dx = \left[x+\frac{x^3}{3}\right]_0^{10}$$
$$= 343\tfrac{1}{3} \text{ shillings}$$

Now this argument begins with an approximation so how can it give the total cost correctly? It succeeds because we have written down, by intuition rather than by reasoning an approximation which is actually a lower rectangle of the graph shown in Fig 5 15 Thus, when we form the sum and take the limit, we obtain the cost exactly When the intuitive solution has been justified for a certain type of calculation, it provides the simplest method of writing down the integrals by which examples of that type may be solved and thus avoids the necessity of learning formulae by heart

The procedure is

(1) Write down a suitable approximation, i e a typical term of the lower or upper sum

(2) Determine the limits between which the summation is to be carried out

(3) Write the limit of the upper or lower sum as an integral

(4) Evaluate the integral

We see from this example that in solving the problem of finding an area with a boundary which is partly curved we have found a new tool with much wider applications This tool is the integral, and the principal object of the remainder of this chapter is to show as many of its applications as we can In each case we shall begin by giving the intuitive solution of the problem and this will then be justified by one of the methods explained in this section

EXAMPLE The acceleration of a body at $t$ sec is $(1+2t/5)$ f p s$^2$ Find (a) the increase of velocity from $t = 1$ to $t = 6$, (b) the velocity at $t = 6$ if the velocity at $t = 1$ is 8 f p s

Consider the short interval of time from $t$ to $t+\delta t$ sec during which the acceleration changes from $(1+2t/5)$ f p s$^2$ to $[1+2(t+\delta t)/5]$ f p s$^2$ If the acceleration is held constant during this interval at its value at the beginning, the increase in velocity from $t$ to $t+\delta t$ sec is given approximately by $(1+2t/5)\,\delta t$ f p s We require the limit of the sum of such increases from $t = 1$ to $t = 6$ as the number of time intervals increases beyond any number, however large, and $\delta t \to 0$

This is
$$\int_1^6 \left(1+\frac{2t}{5}\right) dt = \left[t+\frac{t^2}{5}\right]_1^6$$
$$=\left(6+\frac{36}{5}\right)-\left(1+\frac{1}{5}\right) = 12.$$

(*a*) The increase of velocity is 12 f.p.s.

(*b*) The velocity at $t = 6$ is $8+12 = 20$ f.p.s.

### Exercise 5.F

1. If a contractor estimates that the cost of excavating earth from a depth of $x$ yd. is $(2+10x)$ shillings per cu. yd., what should be his estimate of the total cost of digging a well 12 yd. deep with a cross-section of 2 sq. yd.?

2. A straight road climbs a hill-side and at a horizontal distance of $x$ ft. from a point $A$ at the bottom of the hill the gradient of the road is $x/1000$ (graph gradient). Find the height of the road above $A$ at a horizontal distance of 500 ft. from $A$.

3. A body starts from rest and after $t$ sec. its velocity is $(100-t^2/5)$ f.p.s. Find the distance travelled in 10 sec.

4. An aeroplane climbs at the rate of $(40t+3t^2)$ ft. per min. at $t$ min. Find the increase in its height from $t = 0$ to $t = 10$.

5. A river bursts its banks at $t = 0$ and $t$ hr. later the rate at which the water covers the surrounding land is $100/t^2$ sq. miles per hour $(t > 1)$. Find the area covered during the 24 hours from $t = 1$ to $t = 25$. Find the area covered from $t = 13$ to $t = 25$.

6. A ship springs a leak and after $t$ hr. the water rises in the hold at the rate of $(30-0{\cdot}6t^2)$ in. per hr. What is the depth of water in the hold after 6 hr.?

7. A bore-hole is sunk and it takes $(1+x)/100$ hr. to sink the bore 1 ft. when its total depth is $x$ ft. How long will it take to sink the bore to a depth of 200 ft.?

## 5.9. Volume of a solid of revolution

Let the portion of the curve $y = x^3$ between $x = 1$ and $x = 2$ rotate through 360° about the $x$-axis. The curve sweeps out the surface of a solid which has the $x$-axis as an axis of symmetry. Such a solid is called a *solid of revolution*. Its distinguishing

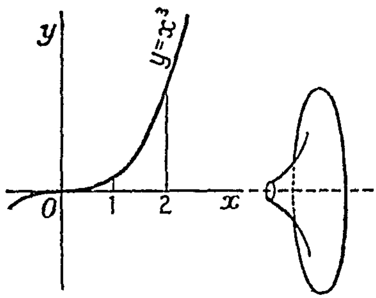


Fig. 5.16

property is that every section perpendicular to the axis of symmetry is a circle. The curve $y = x^3$ is said to *generate*

the solid and is sometimes called the *meridian* curve (The rotation of a meridian about the axis of the earth generates the earth )

Solids of revolution are common in everyday life, e g cylinder, cone, sphere, and all the articles made on a potter's wheel

To find the volume generated by the revolution of $y = x^3$ from $x = 1$ to $x = 2$ we proceed as follows

*Intuitive solution*

Let $P \equiv (x, y)$ be any point on the generating curve such that $x$ is between 1 and 2 In Fig 5 17, $OH = x$ Let

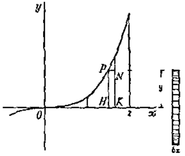



Fig 5 17

$OK = x+\delta x$ and complete the rectangle $PHKN$ The rotation of this rectangle about the $x$ axis generates a circular disk of radius $y$ and thickness $\delta x$

Hence the volume generated by the rotation of the rectangle is $\pi y^2 \, \delta x$

Intuition suggests that the required volume may be found by summing such volumes from $x = 1$ to $x = 2$ and taking the limit as the number of disks increases beyond any number, however large, and $\delta x \to 0$ This limit is

$$\int_1^2 \pi y^2 \, dx = \int_1^2 \pi x^6 \, dx$$

$$= \left[\pi \frac{x^7}{7}\right]_1^2$$

$$= \pi\left(\frac{2^7}{7} - \frac{1}{7}\right)$$

$$= \frac{127\pi}{7}$$

*Justification of the intuitive solution*

*First method.* Let $V$ be the volume generated by the rotation of the portion of the curve between the fixed boundary $x = 1$ and the variable boundary $x = x$.

If $x$ is increased to $x+\delta x$, the increase of $V$ may be represented by $\delta V$ and is the volume generated by the shaded strip in Fig. 5.18 ($a$). This volume lies between the volumes of the two disks shown in Fig. 5.18 ($b$). The thickness of these disks is $\delta x$, the

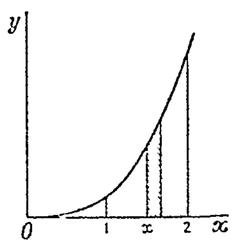



FIG. 5.18 ($a$)

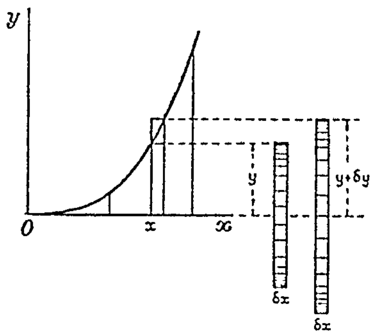



FIG. 5.18 ($b$)

width of the strip, and their radii are $y$ and $y+\delta y$, the least and greatest heights of the strip.

Therefore $$\pi y^2\,\delta x < \delta V < \pi(y+\delta y)^2\,\delta x. \tag{1}$$

Dividing by $\delta x$,

$$\pi y^2 < \frac{\delta V}{\delta x} < \pi(y+\delta y)^2.$$

Let $\delta x \to 0$. $\pi y^2$ is fixed and $\pi(y+\delta y)^2 \to \pi y^2$, while $\dfrac{\delta V}{\delta x} \to \dfrac{dV}{dx}$.

Hence $\dfrac{dV}{dx} = \pi y^2 = \pi x^6$ (since $y = x^3$).

Hence $V = (\pi x^7)/7+c$, where $c$ is a constant.

But $V = 0$ when $x = 1$.

Therefore $0 = \pi/7+c$; so that $c = -\pi/7$

and $V = \pi x^7/7-\pi/7$.

The required volume is found by putting $x = 2$ and is

$$\left(\pi\frac{2^7}{7}-\frac{\pi}{7}\right) = \int_1^2 \pi\, x^6\, dx.$$

*Second method* Draw the graph of $z = \pi y^2 = \pi x^6$ Then $\pi y^2\,\delta x$ and $\pi(y+\delta y)^2\,\delta x$ are $z\delta x$ and $(z+\delta z)\,\delta x$ and are therefore typical lower and upper rectangles for this curve Therefore the lower and upper sums formed from $x = 1$ to $x = 2$ tend to the same limit as the number of rectangles increases beyond any number, however large, and $\delta x \to 0$ But by applying the inequality (1) to each strip from $x = 1$ to $x = 2$ we see that the required volume is between the lower and upper sums Therefore it equals their common limit which is $\int_1^2 \pi y^2\,dx$

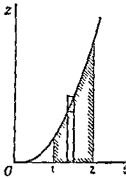



Fig 5 19

EXERCISE 5 G

1 Find the volume generated by revolving the curve $y = 2x^3$ about the $x$ axis from $x = 1$ to $x = 2$

2 Find the volume generated by revolving the curve $y = 5x^2$ about the $x$ axis from $x = 0$ to $x = 1$

3 Find the volume generated by revolving the line $y = x$ about the $x$ axis from $x = 1$ to $x = 4$

4 Find the volume of the solid of revolution generated by the rotation of the part of the curve $y = 10/x$ between $x = 1$ and $x = 5$ about the $x$ axis

5 A cone is generated by rotating $y = \frac{1}{2}x$ from $x = 0$ to $x = 10$ about the $x$ axis Find its volume

6 A cone of height $h$ and radius of base $r$, with its vertex at the origin $O$ and its axis along $Ox$, is generated by rotating a straight line about $Ox$ What must be the equation of the line? Hence show that the volume of the cone is $\frac{1}{3}\pi r^2 h$ [This is the formula of elementary mensuration for the volume of a cone]

7 A vase is obtained by rotating the curve $y = x/4 + 4/x$ from $x = 1$ to $x = 12$ about the axis of $x$, unit $x$ and $y$ being 1 in Make a half scale drawing of the vase If the base of the vase is at $x = 1$ and the top at $x = 12$, what is the diameter of (a) the base, (b) the top? What is the height of the vase? Find the volume of the vase

8 A vase is formed by the rotation of the curve $y^2 = 2x+9$ about the axis of $x$ from $x = 0$ to $x = 8$ the base of the vase being the section at $x = 0$ (unit $x, y = 1$ in ) Find the volume of the vase in cu in

9. The pedestal for a chessman is obtained by rotating

$$y^2 = \frac{1+2x^3}{40}$$

about the $x$-axis from $x = 1$ to $x = 2$ (unit $x$ and $y = 1$ in.). Find the volume of the pedestal.

10. Find the volume of the pointed solid of revolution obtained by rotating about the $x$-axis the portion of the curve $y = x(1-x)$ which lies between $x = 0$ and $x = 1$.

## 5.10. Volumes generated by rotation about the $y$-axis

Consider the solid generated by the rotation of $y = x^2$ about the $y$-axis and suppose we wish to calculate the volume between $y = 0$ and $y = 4$.

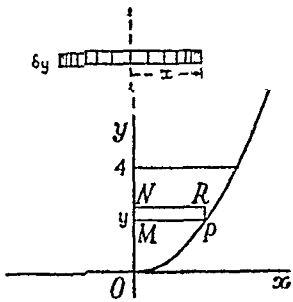


FIG. 5.20

Let $P \equiv (x, y)$ be a point of the generating curve such that $y$ is between 0 and 4. As the rotation is now about $Oy$ it is natural to let $y$ increase by $\delta y$, and consider the rectangle $PMNR$ which generates a disk of radius $x$ and thickness $\delta y$. The volume of this disk is $\pi x^2\, \delta y$ and the required volume is the limit of the sum of such volumes from $y = 0$ to $y = 4$ as the number of disks increases beyond any number, however large, and $\delta x \to 0$. This limit is

$$\int_0^4 \pi x^2\, dy.$$

But $(x, y)$ is a point of the curve $y = x^2$.

Therefore
$$\int_0^4 \pi x^2\, dy = \int_0^4 \pi y\, dy$$
$$= \left[\pi \frac{y^2}{2}\right]_0^4 = 8\pi.$$

EXERCISE 5.H

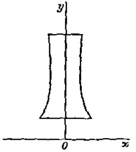
1. Find the volume of the solid formed by rotating the part of the curve $y = 4x^2$ between $y = 1$ and $y = 4$ about the $y$-axis.

2. The part of the curve $4y = x^2$ between $x = 2$ and $x = 8$ is rotated about the $y$-axis. Find the volume generated.

3. A bowl has a circular base of diameter 6 in. and a curved side given by the rotation of $y = (x^2-9)/4$ about the axis of $y$, where $x$ and $y$ are in inches. The top of the bowl has a diameter of 10 in. Sketch the bowl. What is its height? Find its volume.

4 A cup has a circular base of radius 1 in and a side given by the rotation of $y = x^2-1$ about the $y$ axis between $x = 1$ and $x = 2$ (unit $x$ and $y = 1$ in) The saucer has a flat base of radius 1 in to fit the cup and a side formed by the rotation of $y = (x^2-1)/8$ about the $y$ axis from $x = 1$ to $x = 3$ What is the height of (a) the cup, (b) the saucer? Sketch the cup and saucer Find the volume of the cup and of the saucer

A man cools his last cup of tea before catching the train by pouring it into the saucer How many saucersful are obtained from one cup?




5 The part of the circle $x^2+y^2 = a^2$ in the first quadrant is rotated about the $x$ axis to form a hemisphere Find the volume of the hemisphere Deduce a formula for the volume of a sphere of radius $a$

6 The curved boundary of a vase is given by the rotation of $10x^2 = y+14/y^2$ between $y = 1$ and $y = 5$ about the $y$ axis (unit $x$ and unit $y = 1$ in) Find its volume

## 5.11. Volumes generated by rotation about a line parallel to one of the axes

EXAMPLE Find the volume of the solid generated by the revolution of $y = 1-x^2$ from $x = -1$ to $x = 1$ about the line $y = 2$

*Solution* Sketch the curve and line

If $P \equiv (x, y)$ is any point on the part of the curve to be revolved, consider the rectangle $PQRS$ where $PQ = \delta x$, $PS = 2-y$ The volume generated by the revolution of this rectangle about $y = 2$ is $\pi(2-y)^2\,\delta x$ The required volume is the limit of the sum of all such volumes from $x = -1$ to $x = +1$ as their number increases beyond any number, however large, and $\delta x \to 0$ It is therefore given by

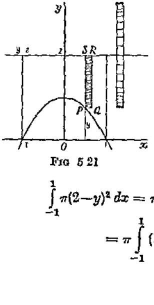



FIG 5 21

$$\int_{-1}^{1} \pi(2-y)^2\,dx = \pi\int_{-1}^{1}(2-1+x^2)^2\,dx = \pi\int_{-1}^{1}(1+x^2)^2\,dx$$

$$= \pi\int_{-1}^{1}(1+2x^2+x^4)\,dx = \pi\left[x+\frac{2}{3}x^3+\frac{x^5}{5}\right]_{-1}^{1}$$

$$= 2\pi(1+\tfrac{2}{3}+\tfrac{1}{5})$$

$$= \frac{56\pi}{15}$$

EXERCISE 5.J

1–6. The part of the curve defined, is rotated about the given line. Find the volume generated.

1. $y = x^2$, from $x = -1$ to $x = +1$, about $y = 1$.
2. $y = 3x$, from $x = 0$ to $x = 2$, about $y = 6$.
3. $y = 2x$, from $x = 0$ to $x = 1$, about $x = 1$.
4. $y^2 = x$, from $y = -3$ to $y = +3$, about $x+1 = 0$.
5. $y = x^3$, from $x = -1$ to $x = 1$, about $y+1 = 0$.
6. $y^2 = 2x$, from $y = -2$ to $y = 2$, about $x = 2$.
7. Find the equation of the tangent at $x = 0$ to $y = 4-x^2$. Find the volume generated by the rotation of the part of the curve between $x = -2$ and $x = 2$ about this tangent.
8. Find the ratio of the volume obtained by rotating $y = x^2$ from $(0, 0)$ to $(1, 1)$ about $y = 1$ to the volume obtained by rotating it about $y+1 = 0$.

## 5.12. The volume of a pyramid

If the vertices of a polygon with any number of sides are joined by straight lines to a point $V$, not in the plane of the polygon, the solid, so constructed, is called a *pyramid*. $V$ is called the *vertex* of the pyramid and the polygon is called its *base*. A perpendicular drawn from the vertex to the base of the pyramid is called its *height*.

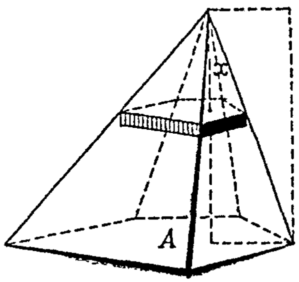



FIG. 5.22

Fig. 5.22 shows a pyramid whose base is a pentagon of area $A$. The height of the pyramid is $h$. To find its volume, consider the section made by a plane parallel to the base at a perpendicular distance $x$ from the vertex. The section and the base are similar polygons and the ratio of corresponding dimensions is $x:h$. Hence the area of the section is $A(x/h)^2$. On the section construct a thin solid of uniform thickness $\delta x$ measured at right angles to the section. The volume of this solid is $(Ax^2\,\delta x)/h^2$.

Let other thin solids be constructed, in the same way, so that their combined thickness reaches from the vertex to the base, and together they occupy the whole volume of the pyramid except for the narrow 'steps' near its sloping sides (Fig 5 23) The sum of the volumes of these solids is a little less than the volume of the pyramid Assuming that it is a lower sum, so that it tends to the volume of the pyramid as a limit when the number of solids is increased beyond any number, however large, and the thickness of each is decreased, we have for the volume of the pyramid

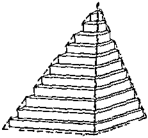


FIG 5 23

$$\int_0^h \frac{Ax^2}{h^2}\,dx = \frac{A}{h^2}\int_0^h x^2\,dx$$

$$= \frac{A}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{Ah^3}{3h^2} = \frac{Ah}{3}.$$

This result is independent of the shape of the base of the pyramid If the base is a smooth curve, the solid is called a *cone* If the curve is a circle it is called an *oblique circular cone*, and if the straight line from the centre of the circle to the vertex of the cone is perpendicular to the base it is called a *right circular cone*, or in elementary work, simply a cone The right circular cone is a solid of revolution and its volume has already been found in Ex 5 G, No 6 Here we have proved the more general result that the volume of a cone, right or oblique, is $\frac{1}{3}Ah$ where $A$ is the base area and $h$ the perpendicular height

EXERCISE 5.K

Trace out the argument, independently, for finding the volume of a right circular cone of height $h$ and base radius $r$

## 5.13. The mass of a body of variable density

In mechanics it is important to distinguish between *mass* and *weight* As the next application of integration is to mechanics, we first consider this distinction very briefly

The weight of a body is the pull of the earth upon it It is therefore a force

A man enters a grocer's shop to buy 1 lb. of butter. The grocer measures the quantity of butter by the process known as weighing. He adds or subtracts butter until the pull of the earth upon it equals the pull on a brass weight in the other scale pan. That is to say, he utilizes weight to measure the quantity of butter which the customer wishes to buy.

When we wish to indicate that 1 lb. is thought of as a measure of quantity, we call it the *mass* of the body.

A housewife buys 12 lb. of potatoes. When she gives the order, she is thinking of the quantity she requires, i.e. she thinks of 12 lb. as the mass of the potatoes. But as she carries them home, her arm aches, and she thinks, 'Why did I order so many?' What she really means is, 'Why did I not think of their weight as well as of their mass?'

The pull of the earth upon a mass of 1 lb. is a force which is called 1 lb. weight. This force provides a unit for the measurement of all forces including weight. Thus the weight of a mass of $x$ lb. is a force of $x$ lb. wt.

Let $m$ be the mass of a volume $v$ of a body. If the ratio $m/v$ is found to be a constant whatever volume of the body is considered, we say that the body is *uniform* or that it has *constant density*. In this case the *density* of the solid is $d$, where $d = m/v$. If $v = 1$, $d = m$. Hence, the density, if it is constant, is the mass of unit volume. Thus the unit of measurement of constant density is mass per unit volume, e.g. the density of pine wood is given as 50 lb. per cu. ft. The density of a given material is commonly used to estimate the mass of a given volume of the material. For example, we estimate the mass of a pine beam whose volume is $2\frac{1}{2}$ cu. ft. as $50 \times 2\frac{1}{2} = 125$ lb.

If the density of the body is not constant this will show itself in a variation of the value of $m/v$ when different volumes are taken. In this case $m/v$ is called the *average density* of the volume $v$ of the body. Consider a definite point $P$ of the body and let $\delta v$ be a small volume of the body which has $P$ inside it. Let $\delta m$ be the mass of this volume. Now let the volume $\delta v$ tend to zero in such a way that $P$ is always inside it. Then if $\delta m/\delta v$ tends to a limit, this limit is the *density* of the body *at P*.

Suppose we want to consider a uniform solid made of iron in the form of a flat plate of constant thickness $\frac{1}{16}$ in. If the density of the iron is 480 lb. per cu. ft. and the area of the plate is $a$ sq. ft., the mass of the plate is

$$480 \times a \times \frac{1}{16 \times 12} = \frac{5a}{2} \text{ lb.}$$

Hence the mass is found by multiplying the area by $2\frac{1}{2}$.

Clearly, each square foot of the plate weighs $2\frac{1}{2}$ lb and as long as we consider iron plates of thickness $\frac{1}{16}$ in this *mass per unit area* is more convenient for the calculation of the mass of a given plate than the density of iron

Hence when we are considering a solid in the form of a flat plate we often give the mass per unit area instead of the density

Similarly when we want to calculate the mass of a rod of given thickness it is more convenient to know the *mass per unit length* than the density of the material of which the body is made

For example, in an engineering note book it is stated that the mass per unit length of round steel bars of $\frac{3}{4}$ in diameter is $1\frac{1}{2}$ lb per ft Then the mass of such a bar 5 ft long is $5\times 1\frac{1}{2} = 7\frac{1}{2}$ lb

When the density of a body is constant, the calculation of the mass of a given volume is a simple multiplication But if the density is variable, the calculation requires the use of an integral, and it is the simpler calculations of this type which interest us here

EXAMPLE 1 The mass per unit length of a straight rod $OA$, 4 ft long, at a distance $x$ ft from $O$ is $(1+0\ 6x^2)$ lb per ft Find the mass of the rod


FIG 5 24

If $OP = x$ and $OQ = (x+\delta x)$ ft, we call the short length, $PQ$, an *element* of the rod

Its mass per unit length varies from $(1+0\ 6x^2)$ lb per ft at $P$ to $[1+0\ 6(x+\delta x)^2]$ lb per ft at $Q$

*Intuitive solution* By holding the mass per unit length of the element constant at its value at $P$, we obtain $(1+0\ 6x^2)\,\delta x$ lb as an approximation to its mass Assuming that this is a suitable approximation, the mass of the rod $OA$ is the limit of the sum of such approximations for all the elements of the rod as the number of elements increases beyond any number, however large, and $\delta x \to 0$ This limit is

$$\int_0^4 (1+0\ 6x^2)\,dx = \left[x+\frac{0\ 6x^3}{3}\right]_0^4$$
$$= 4+0\ 2\times 4^3$$
$$= 16\ 8$$

Therefore, the mass of the rod is 16 8 lb

*Justification of the intuitive solution.* The smallest mass per unit length of the element, $PQ$, is $(1+0{\cdot}6x^2)$ lb. per ft. and the largest mass per unit length is $[1+0{\cdot}6(x+\delta x)^2]$ lb. per ft. Hence the mass of the element is greater than $(1+0{\cdot}6x^2)\,\delta x$ lb. and less than $[1+0{\cdot}6(x+\delta x)^2]\,\delta x$ lb.

By writing $y = 1+0{\cdot}6x^2$ these expressions simplify to $y\,\delta x$ and $(y+\delta y)\,\delta x$ and are then recognized as lower and upper rectangles of the graph of

$$y = 1+0{\cdot}6x^2.$$

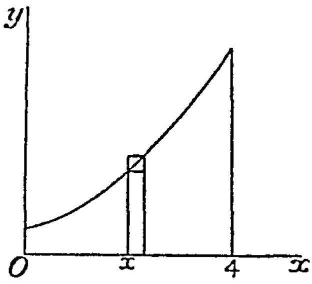


FIG. 5.25

Therefore the lower and upper sums, from $x = 0$ to $x = 4$, have the same limit as the number of rectangles is increased beyond any number, however large, and $\delta x \to 0$. This limit is $\int_0^4 y\,dx$. But the mass of the rod is known to lie between the lower and upper sums and is, therefore, equal to their common limit.

Hence $\int_0^4 y\,dx = \int_0^4 (1+0{\cdot}6x^2)\,dx$, the integral used in the intuitive solution, gives the mass of the rod exactly.

EXAMPLE 2. Owing to the greater compression near the base, the density of a conical mound of earth at a depth $x$ ft. is $(80+x/6)$ lb. per cu. ft. If the radius of the base is 6 ft. and the height of the vertex is 4 ft., find the mass of the mound.

*Intuitive solution.* Fig. 5.26 shows the vertical cross-section of the mound. Consider the layer of earth at a depth between $x$ ft. and $(x+\delta x)$ ft., and let the radius of the horizontal cross-section of the mound at depth $x$ ft. be $r$ ft. The volume of this layer is approximately the volume of a disk of radius $r$, and its density varies from $(80+x/6)$ lb. per cu. ft. at the top to $[80+(x+\delta x)/6]$ lb. per cu. ft. at the bottom. Taking the density of the layer as constant at $(80+x/6)$ lb. per cu. ft., we find that the mass of the layer is approximately $\pi r^2\,\delta x(80+x/6)$ lb. If we assume, intuitively, that the mass of the mound is the limit of the sum of such approximations from $x = 0$ to $x = 4$, as the number of layers is increased beyond any number, however large, and $\delta x \to 0$,

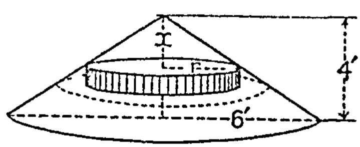


FIG. 5.26

$$\text{the whole mass} = \int_0^4 \pi r^2(80+x/6)\,dx.$$

Also by similar triangles

$$\frac{r}{x} = \frac{6}{4} = \frac{3}{2}$$

$$r = \frac{3x}{2}$$

$$\text{Hence the mass} = \int_0^4 \pi\left(\frac{3x}{2}\right)^2\left(80+\frac{x}{6}\right)dx$$

$$= \int_0^4 9\frac{\pi}{4}\left(80x^2+\frac{x^3}{6}\right)dx$$

$$= 9\frac{\pi}{4}\left[\frac{80x^3}{3}+\frac{x^4}{24}\right]_0^4$$

$$= \pi(3840+24)$$

$$= 3864\pi \text{ lb}$$

$$= 5\ 4 \text{ tons}$$

*Justification of the intuitive solution*


FIG 5 27

Let the mass of the mound down to a depth $x$ ft be $M$ lb Then the mass of the layer whose depth is between $x$ and $x+\delta x$ ft is $\delta M$ lb Now the volume of this layer is greater than $\pi r^2\,\delta x$ cu ft and its density is greater than $80+x/6$ lb per cu ft Therefore

$$\delta M > \pi r^2\left(80+\frac{x}{6}\right)\delta x$$

Also, the volume of the layer is less than $\pi(r+\delta r)^2\,\delta x$ cu ft and its density is less than $[80+(x+\delta x)/6]$ lb per cu ft

Therefore $$\delta M < \pi(r+\delta r)^2\left(80+\frac{x+\delta x}{6}\right)\delta x$$

Putting these two inequalities together

$$\pi r^2\left(80+\frac{x}{6}\right)\delta x < \delta M < \pi(r+\delta r)^2\left(80+\frac{x+\delta x}{6}\right)\delta x$$

Dividing by $\delta x$,

$$\pi r^2\left(80+\frac{x}{6}\right) < \frac{\delta M}{\delta x} < \pi(r+\delta r)^2\left(80+\frac{x+\delta x}{6}\right)$$

Let $\delta x \to 0$, then $\delta M/\delta x \to dM/dx$
and, since $\delta r \to 0$,

$$\pi(r+\delta r)^2\left(80+\frac{x+\delta x}{6}\right) \to \pi r^2\left(80+\frac{x}{6}\right).$$

Therefore $$\frac{dM}{dx} = \pi r^2\left(80+\frac{x}{6}\right)$$

Therefore, by an argument which is now so familiar that it need not be given in detail, the required mass $= \int_0^4 \pi r^2\left(80+\frac{x}{6}\right) dx$.

This is the integral used in the intuitive solution.

## Exercise 5.L

1. The mass per unit length of a straight rod at a distance of $x$ ft. from one end is $(3-x/2)$ lb. per foot. If the rod is 4 ft. long, find its mass.

2. The mass per unit length of a billiard cue, 4 ft. 6 in. long, is $\frac{1}{3}(4+2x)$ oz. per foot where $x$ is the distance in ft. from one end. Find its mass.

3. A rectangular plate is 4 ft. long and 3 ft. wide. The mass per unit area is constant across the width of the plate, and at a distance $x$ ft. from one 3 ft. edge it is $8(5-x)$ lb. per sq. ft. Find the mass of the plate.

4. A conical mound of earth has the radius of its base equal to its height which is 3 ft. At $x$ ft. below the vertex the density is $(100+x/10)$ lb. per cu. ft. Show that the mound weighs just over $902\pi$ lb.

5. A mound of earth is a pyramid of height 5 ft. and the area of its base is 36 sq. ft. The density at depth $x$ ft. below the vertex is $(100+3x^2/4)$ lb. per cu. ft. Find the mass of the mound.

6. Show that the area of a circular ring is approximately $2\pi r\,\delta r$, where $r$ and $r+\delta r$ are its inner and outer radii.

Find the mass of a circular plate of radius 2 ft., if its mass per unit area at distance $r$ ft. from the centre is $(10+0{\cdot}3r)$ lb. per sq. ft.

7. The density of the air $x$ km. above the earth's surface is given approximately, up to a height of 10 km., by the formula $d=(4x^2-124x+1260)$ gm. per cu. metre. Show that the mass of the air between the heights of $x$ km. and $(x+dx)$ km. in a column of cross-sectional area 1 sq. metre is approximately

$$(4x^2-124x+1260)\,dx\,\text{kg}.$$

Find in kg. to 2 sig. fig. the mass of the column of air up to a height of 10 km.

The mass of the atmosphere above each sq. metre of the earth's surface is known to be about 10,300 kg. What percentage by weight of the air in a column of cross-sectional area 1 sq. metre is at a height greater than 10 km.?

## 5.14. Centre of gravity

If a hole is bored through the middle point of a metre rule and it is then mounted on a horizontal axis passing through the hole, the rule will be found to remain at rest at whatever angle to the vertical it may be placed

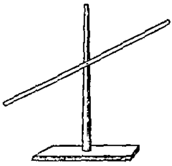


Fig 5 28

The rule is said to *balance* about its middle point

Now suppose the same experiment is performed with a thin tapering rod, e g a billiard cue If *a hole* is bored midway between the two ends, the rod will not balance about this point To make it balance, the hole must be made somewhere between the thick end and the middle point This point of balance is called the *centre of gravity* (abbreviation c g ) of the rod

Since the attraction of the earth, which is really spread over the whole body, is balanced by a single upward force at the c g , we may say that the attraction of the earth on the whole rod can, for the problems of mechanics, be regarded as equivalent to a single downward force acting at the c g

The c g of a rod of constant mass per unit length is the middle point of the axis of the rod Such a rod is called a *uniform* rod

Now consider a uniform metal rod $AC$, 3 ft long, weighing 3 lb Suppose the rod is bent at right angles at a point $B$, 1 ft from $C$ If the angle $B$ is placed over a support, the rod will hang at rest with the arms in a vertical plane Is $B$ the point of balance? No, because if the rod is displaced it will always be found to swing back into one position of rest

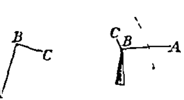



Fig 5 29

*Or, if the rod is supported at B with its arms in a horizontal* plane, the rod will be found to fall

The point $B$ is not the c g because the rod will not remain at rest when it is supported there, *whatever the position in which it is placed*

The part $AB$ may be regarded as a uniform straight rod Its weight, 2 lb , may be taken to act at its c g which is the middle point $M$ of $AB$ (Fig 5 30) Similarly the weight of $BC$, 1 lb , may be regarded as acting at $N$, the middle point of $BC$ Hence the

c.g. of $ABC$ is a point about which weights of 2 lb. at $M$ and 1 lb. at $N$ balance. This point must be somewhere on the straight line $MN$ between $M$ and $N$. This means that the c.g. is not a point of the rod at all, so that it is not possible to balance it on a support. Nevertheless it is still useful to know that there is a point through which the whole weight of the body may be considered to act. We therefore define the centre of gravity of a body as the point of balance regardless of whether the balancing is physically possible or not.† The centre of gravity of a body may or may not be a point inside the material of the body.

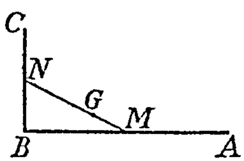



FIG. 5.30

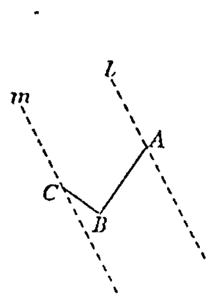



FIG. 5.31

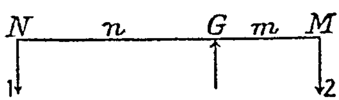



FIG. 5.32

However, if all the material of a body lies between two parallel planes, $l$ and $m$ (Fig. 5.31), it follows that the point of balance also lies between $l$ and $m$, and therefore the c.g. of the body lies between $l$ and $m$. This is an important fact of which frequent use will be made.

Another simple result which we frequently need is that if a body has an axis of symmetry, the c.g. must be a point of this axis. For example, the c.g. of a solid of revolution must be on the axis of rotation.

If $G$ is the c.g. of $ABC$ and $GM = m$, $GN = n$, by the principle of moments‡ we have $1 \times n = 2 \times m$ (Fig. 5.32).

Therefore $m/n = \frac{1}{2}$, or $G$ divides $MN$ in the ratio of $1:2$ (i.e. in the inverse ratio of the weights at $M$ and $N$).

It is useful to find the position of $G$ by a different argument.

† The experiment of balancing the rod $ABC$ at $G$ could actually be performed by first attaching it to a uniform circular disk in such a way that the centre of the disk is at $G$.

‡ The reader who is not acquainted with this principle is referred to any elementary text-book on mechanics.

Place the rod in a horizontal plane, take $B$ as the origin, $BA$ as $x$ axis, and $BC$ as $y$ axis

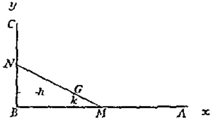



FIG 5 33

Let the coordinates of $G$ be $(h, k)$ Imagine that the rod $ABC$ is supported on an axis coinciding with $By$ Then, if the whole weight of the body may be considered to act at $G$, its turning effect about $By$ equals the combined turning effect of the weight of $BA$ at $M$ and the weight of $BC$ at $N$ Hence taking moments about $By$

$$3h = 2\times 1+1\times 0$$
$$3h = 2$$
$$h = \tfrac{2}{3}$$

To find $k$, take moments about $Bx$

$$3k = 2\times 0+1\times \tfrac{1}{2}$$
$$3k = \tfrac{1}{2}$$
$$k = \tfrac{1}{6}$$

Thus $G \equiv (\tfrac{2}{3}\ \tfrac{1}{6})$ or the c g is 8 in from $BC$ and 2 in from $BA$

Now consider an example in which the determination of the c g requires the use of integrals

EXAMPLE The mass per unit length of a straight rod $OA$, 4 ft long, at a distance of $x$ ft from $O$ is $(1+0\ 6x^2)$ lb per ft Find the position of the c g of the rod


FIG 5 34

In section 5 13 Example 1 we have already found that the mass of the rod is 16 8 lb Denote the position of the c g by $G$ and let $OG = h$ ft Then the weight of the rod is 16 8 lb and may be regarded as acting at $G$ For convenience in taking moments, suppose the rod is horizontal and take moments about a horizontal axis through $O$ at right angles to the rod The moment of the weight, concentrated at $G$, equals the sum of the moments of the weights of all the particles of the rod Before we can write down the equation of moments we must calculate this sum of moments

*Intuitive solution* Consider the element $PQ$, where $OP = x$ ft and $OQ = (x+\delta x)$ ft The weight of the element is approximately $(1+0\ 6x^2)\ \delta x$ lb and its line of action passes through the c g of the

element. Since $PQ$ is not uniform we do not know the position of the c.g. and we approximate still further by taking it to be $P$. The approximate moment of the weight of the element about $O$ is then $(1+0{\cdot}6x^2)\,\delta x\times x$. We now form the sum of such approximations to the moments of all the elements of the rod from $x=0$ to $x=4$, and take the limit as the number of elements increases beyond any number, however large, and $\delta x\to 0$. This limit is $\int_0^4 (1+0{\cdot}6x^2)x\,dx$ and, if the approximations can be justified, it is the required sum of moments.

The equation of moments is, therefore,

$$\begin{aligned}16{\cdot}8h &= \int_0^4 (x+0{\cdot}6x^3)\,dx\\ &= \left[\frac{x^2}{2}+\frac{0{\cdot}6x^4}{4}\right]_0^4\\ &= 8+0{\cdot}6\times 64\\ &= 46{\cdot}4.\end{aligned}$$

Therefore $$h=\frac{46{\cdot}4}{16{\cdot}8}\doteqdot 2{\cdot}8\text{ ft.}$$

*Justification of the intuitive solution.* The weight of the element, $PQ$, is greater than $(1+0{\cdot}6x^2)\,\delta x$ lb. and less than $[1+0{\cdot}6(x+\delta x)^2]\,\delta x$ lb. Also its c.g. is between $P$ and $Q$, by its definition as a point of balance, i.e. its distance from $O$ is greater than $x$ ft. and less than $(x+\delta x)$ ft. Hence the moment of the element about $O$ is greater than the product of the smaller weight and distance and is less than the product of the larger weight and distance. Therefore it is between $(1+0{\cdot}6x^2)x\,\delta x$ and $[1+0{\cdot}6(x+\delta x)^2](x+\delta x)\,\delta x$.

If we write $$y=(1+0{\cdot}6x^2)x,$$
so that $$y+\delta y=[1+0{\cdot}6(x+\delta x)^2](x+\delta x),$$
the moment of the element lies between $y\,\delta x$ and $(y+\delta y)\,\delta x$. These are now recognized as typical lower and upper rectangles of the curve $y=x+0{\cdot}6x^3$ and, therefore, the lower and upper sums formed from them, from $x=0$ to $x=4$, tend to the limit $\int_0^4 (x+0{\cdot}6x^3)\,dx$ as the number of rectangles increases beyond any number, however large, and $\delta x\to 0$.

Therefore the sum of moments, which lies between the lower and upper sums, equals their common limit,

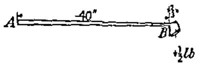
$$\int_0^4 (x+0{\cdot}6x^3)\,dx.$$

EXERCISE 5 M

1 A rod is 1 ft long and its mass per unit length at $x$ ft from one end is $6x$ lb per ft Find the mass of the rod and the distance of the c g from the light end

2 *The mass per unit length of a rod $OA$, 6 ft long, at a distance* $x$ ft from $O$ is $(1+0\ 3x)$ lb per ft Find, to the nearest inch, the distance of its c g from the light end

3 The weight per unit length of the shaft of a golf club is $(\frac{1}{10}+x/200)$ oz per in at $x$ in from $B$ Find the weight of the shaft, $AB$, and the distance of its c g from $B$ Find the distance from $B$ of the c g of the club as a whole




## 5 15 The centre of gravity of a uniform solid of revolution

E Consider the uniform solid formed by rotating the part of the curve $y = x^3$ between (1, 1) and (2 8) about the $x$ axis Since the $x$ axis is the axis of symmetry of the body, the c g is on this axis and may be taken as the point $(h, 0)$ Since the position of the c.g is independent of the position of the body in relation to the earth, let the body be placed so that the axis of symmetry is horizontal and suppose that the $y$ axis is also horizontal Then, Fig 5 35 represents a horizontal section of the body and all weights act in a direction perpendicular to this plane This is convenient for taking moments

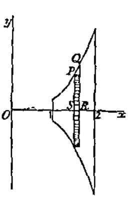



FIG 5 35

We have already found, in section 5 9, that the volume of the solid is $127\pi/7$ Let its density be $m$ lb per unit volume Since the solid is uniform, $m$ is a constant and the weight of the solid is $(127m\pi)/7$lb

*Intuitive solution* In order to write down the equation of moments, we require the sum of the moments of the weights of the particles of the body about $Oy$ Consider the *slice* generated by the rotation of the shaded area $PQRS$, where $OS = x$ and $OR = x+\delta x$ This slice is approximately a disk of radius $PS = y$ and thickness $\delta x$ Its weight is approximately $m\pi y^2\ \delta x$ lb and its c g is a point of $Ox$ between $S$ and $R$ We approximate again by taking the c g at $S$ Then the moment of the slice is approximately $m\pi y^2\ \delta x \times x$ We now form the sum

of such approximations to the moments of all the slices of the body from $x = 1$ to $x = 2$. The limit of this sum, as the number of slices increases beyond any number, however large, and $\delta x \to 0$, is $\int_1^2 m\pi x y^2\,dx$. Assuming that the approximations are justified, the required sum of moments is

$$\begin{aligned}&\int_1^2 m\pi x y^2\,dx\\ &= \int_1^2 m\pi x^7\,dx\\ &= m\pi\left[\frac{x^8}{8}\right]_1^2\\ &= \frac{255m\pi}{8}.\end{aligned}$$

Hence the equation of moments is

$$\frac{127m\pi}{7}\times h = \frac{255m\pi}{8}$$

$$h = \frac{255\times 7}{127\times 8}$$

$$\doteqdot 1{\cdot}76.$$

*Justification of the intuitive solution*

Let the moment about the $y$-axis of the weight of the part of the body between $x = 1$ and $x = x$ be denoted by $M_y$. Then the moment of the slice generated by the rotation of $PQRS$ is $\delta M_y$. The volume of the slice is greater than the volume generated by the rotation of the rectangle $PMRS$ and less than the volume generated by the rotation of the rectangle $NQRS$.

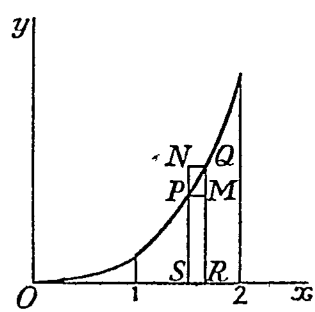


FIG. 5.36

Therefore its weight is greater than $m\pi y^2\,\delta x$ and less than $m\pi(y+\delta y)^2\,\delta x$.

Now the moment of the slice may be found by regarding its whole weight as acting at its c.g. which is a point on $Ox$ between $S$ and $R$, by its definition as a point of balance. Hence $\delta M_y$ is greater than $m\pi y^2\,\delta x\times x$ and less than $m\pi(y+\delta y)^2\,\delta x\times(x+\delta x)$. Therefore $m\pi y^2 x\,\delta x < \delta M_y < m\pi(y+\delta y)^2\,(x+\delta x)\,\delta x$.

Dividing by $\delta x$

$$m\pi y^2 x < \frac{\delta M_y}{\delta x} < m\pi(y+\delta y)^2(x+\delta x)$$

Let $\delta x \to 0$ Then $m\pi(y+\delta y)^2(x+\delta x) \to m\pi y^2 x$

and $$\frac{\delta M_y}{\delta x} \to \frac{dM_y}{dx}$$

Therefore $$\frac{dM_y}{dx} = m\pi y^2 x,$$

from which we deduce, in the usual way, that the required sum of moments is exactly $\int_1^2 m\pi y^2 x\, dx$

EXERCISE 5.N

1–4 Find the position of the c g of the uniform solids of revolution generated by the revolution of the curves given

1 The part of the curve $y = x^2$ from (0, 0) to (2, 4) rotated about the $x$ axis

2 The part of the curve $y = 1/x^2$ from (1, 1) to $(2, \frac{1}{4})$ rotated about the $x$ axis

3 The part of the line $y = 2x$ from (0, 0) to (4, 8) rotated about the $x$ axis

4 The part of the curve $3y = x^2$ from (0, 0) to (3, 3) rotated about the $y$ axis

5 A uniform solid cone is obtained by rotating $y = \frac{1}{2}x$ from $x = 0$ to $x = 4$ about the $x$ axis Find the position of its c g

6 Any uniform solid cone† may be obtained by rotating $y = mx$ about the $x$ axis, if $m$ is given a suitable value If the height of the cone is $h$, we need only consider values of $x$ between 0 and $h$ By rotating the portion of $y = mx$ between $x = 0$ and $x = h$ about the $x$ axis, prove that the distance from the vertex of the c g of a cone of height $h$ is $\frac{3}{4}h$

7 A uniform solid is generated by the revolution of the part of the curve $y^2 = 16x$ between $x = 0$ and $x = h$ about the $x$ axis Find the coordinates of the c g of the solid

8 By revolving a quarter of the circle $x^2+y^2 = a^2$ about the $x$ axis show that the c g of a uniform solid hemisphere of radius $a$ is $3a/8$ from the centre of the base on the perpendicular to the base through the centre

9 Find the c g of the uniform solid lying between $y = 0$ and $y = k$ and generated by rotating part of the curve $ay = x^2$ about the $y$ axis Show that its position depends on $k$ and not on $a$

† i e any right circular cone

10. Find the c.g. of the uniform solid generated by the rotation of the part of the curve $y = x(2-x)$ from $(0, 0)$ to $(1, 1)$ about the $x$-axis.

## 5.16. The centre of gravity of a uniform lamina

A *lamina* is a solid in the form of a plane sheet of small and constant thickness. If the density is also constant it is called a uniform lamina and in this section we show how to find the c.g. of such a lamina when part of the boundary, at least, is curved. In certain simple cases no calculation is necessary, e.g. the c.g. of a uniform rectangular lamina is at the intersection of its diagonals; the c.g. of a uniform circular lamina is at its centre.

Consider the lamina bounded by

$$x = 0,\ y = 0,\ y = 4-x^2,$$

and let its mass per unit area be $m$, where $m$ is constant. Denote the c.g. by the point $G \equiv (h, k)$.

Since the position of the c.g. is independent of the position of the body relative to the earth, we assume that the plane of the lamina is horizontal, so that all weights act at right angles to the plane of Fig. 5.37.

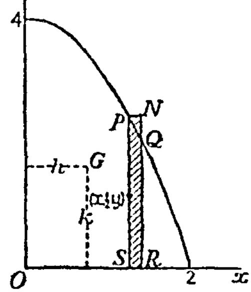


Fig. 5.37

*Intuitive solution.* The weight of the whole body, being $m$ times its area, is $m \int_0^2 y\, dx$.

To find $h$ we take moments about the $y$-axis, equating the moment of the whole weight acting at $G$ to the sum of the moments of the weights of all the particles in the body. Since this principle applies to any part of the body, the contribution to the sum of moments made by the particles of the strip $PQRS$, where $OS = x$ and $OR = x+\delta x$, is the moment of the weight of the strip acting at its c.g. But $PQRS$ coincides approximately with the rectangle $PNRS$, so that its weight is approximately $my\,\delta x$ and its c.g. is near the point $(x+\frac{1}{2}\delta x,\ \frac{1}{2}y)$, the intersection of the diagonals of the rectangle. If we approximate again by taking the c.g. to be the point $(x, \frac{1}{2}y)$, the middle point of $PS$, we have the approximation $my\,\delta x \times x$ for the moment of the strip. The required sum of moments for the whole body is the limit of the sum of such approximations for all strips of the body from $x = 0$ to $x = 2$ as the number of strips is increased beyond any number, however large, and $\delta x \to 0$,

provided the approximations can be justified This limit is

$$\int_0^2 mxy\,dx$$

Therefore the equation of moments is

$$h\times m\int_0^2 y\,dx = \int_0^2 mxy\,dx$$

$$= m\int_0^2 xy\,dx,$$

since $m$ is constant

Also $P \equiv (x, y)$ is on the curve $y = 4-x^2$

Therefore $$h\int_0^2 (4-x^2)\,dx = \int_0^2 (4x-x^3)\,dx$$

$$h\left[4x-\frac{x^3}{3}\right]_0^2 = \left[2x^2-\frac{x^4}{4}\right]_0^2$$

$$h\left(8-\frac{8}{3}\right) = \left(8-\frac{16}{4}\right)$$

$$\frac{16h}{3} = \frac{16}{4}$$

$$h = \tfrac{3}{4}$$

To find $k$ we take moments about the $x$ axis, making the same approximations as before The moment of the weight of the strip about $Ox$ is approximately $my\,\delta x\times\frac{1}{2}y$, and this leads to $\int_0^2 \frac{1}{2}my^2\,dx$ as the exact value of the sum of moments, provided that the approximations can be justified

The equation of moments about $Ox$ is

$$km\int_0^2 y\,dx = \int_0^2 \tfrac{1}{2}my^2\,dx = \frac{m}{2}\int_0^2 y^2\,dx \qquad (1)$$

$$\int_0^2 y\,dx = \frac{16}{3} \text{ as before,}$$

$$\int_0^2 y^2\,dx = \int_0^2 (16-8x^2+x^4)\,dx$$

$$= \left[16x-\frac{8}{3}x^3+\frac{x^5}{5}\right]_0^2$$

$$= 32-\frac{64}{3}+\frac{32}{5} = \frac{256}{15}$$

Therefore from equation (1), $\dfrac{16k}{3} = \dfrac{128}{15}$

$$k = \frac{8}{5}.$$

The c.g. of the lamina is the point $(\frac{3}{4}, 1\frac{3}{5})$.

*Justification of the intuitive solution.* Let the area of the part of the lamina between the $y$-axis and $SP$ be denoted by $A$. Then the area of the strip $PQRS$ may be represented by $\delta A$. The moment of this strip about the $y$-axis is its weight $m\,\delta A$ multiplied by the distance of its c.g. from $Oy$. Now this c.g., the point $C$ in Fig. 5.38, lies somewhere between $PS$ and $QR$ by its definition as a point of balance and so the moment of the strip is greater than $m\,\delta A.x$. Since $\delta A$ is greater than $(y+\delta y)\,\delta x$, the moment of the strip is greater than $m(y+\delta y)\,\delta x.x$. Similarly the moment of the strip is less than $m\,\delta A(x+\delta x)$ and $\delta A$ is less than $y\,\delta x$, so that the moment is less than $my\,\delta x(x+\delta x)$.

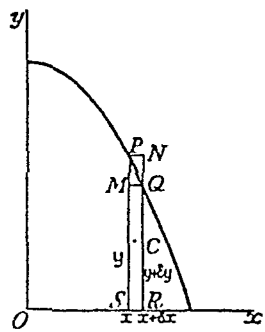



FIG. 5.38

Then, if $M_y$ is the moment about the $y$-axis of that part of the lamina between $Oy$ and $PS$ and $M_y+\delta M_y$ the moment of that part of the lamina between $Oy$ and $QR$, $\delta M_y$ is the moment of the strip and

$$mx(y+\delta y)\,\delta x < \delta M_y < my(x+\delta x)\,\delta x,$$

or
$$mx(y+\delta y) < \frac{\delta M_y}{\delta x} < my(x+\delta x).$$

Let $\delta x \to 0$. Then $mx(y+\delta y) \to mxy$ and $my(x+\delta x) \to mxy$.

Therefore
$$\frac{dM_y}{dx} = mxy$$

and the sum of moments about the $y$-axis $= \int\limits_0^2 mxy\,dx$ exactly.

This is the integral used in the intuitive solution.

Now let $M_x$ be the moment about the $x$-axis of the part of the lamina between $Oy$ and $PS$ and $M_x+\delta M_x$ the moment of the part between $Oy$ and $QR$, so that $\delta M_x$ is the moment of the strip. By the definition of the c.g. as a point of balance, $C$, the c.g. of the strip $PQRS$, is nearer the $x$-axis than $(x+\frac{1}{2}\delta x, \frac{1}{2}y)$, the c.g. of the rectangle $PNRS$. Therefore the moment of the strip

about the $x$ axis is less than $m\,\delta A\times\frac{1}{2}y$, which is less than $my\,\delta x\times\frac{1}{2}y$, since $\delta A$ is less than $y\,\delta x$ Hence $\delta M_x < \frac{1}{2}my^2\,\delta x$

Similarly, $C$ is farther from the $x$ axis than $(x+\frac{1}{2}\delta x, \frac{1}{2}(y+\delta y))$, the c g of the rectangle $MQRS$

Therefore $\delta M_x > m\,\delta A\times\frac{1}{2}(y+\delta y) > m(y+\delta y)\,\delta x\times\frac{1}{2}(y+\delta y)$

Putting the two inequalities together and dividing by $\delta x$, we have

$$\tfrac{1}{2}m(y+\delta y)^2<\frac{\delta M_x}{\delta x}<\tfrac{1}{2}my^2$$

When $\delta x\to 0$, $\frac{1}{2}m(y+\delta y)^2\to\frac{1}{2}my^2$

Therefore $$\frac{dM_x}{dx}=\tfrac{1}{2}my^2,$$

and the sum of moments about the $x$ axis $=\int_0^2 \frac{1}{2}my^2\,dx$ exactly

This is the integral used in the intuitive solution.

### Exercise 5 P

1–8 Find the position of the c g of the uniform lamina bounded by the loci given

1 $y=x^2$, $y=0$, $x=2$ 2 $y=4x^3$, $x=1$, $y=0$

3 $y=4-x^2$, $y=0$

4 $x=0$, $y=0$ and the part of $y=1-x^2$ for which $x$ and $y$ are both positive

5 $y=10x(1-x)$, $y=0$

6 (i) $y=2x$, $y=0$, $x=6$ (ii) $y=mx$, $y=0$, $x=h$

7 $y=3x^2$, $x=1$, $x=2$, $y=0$

8 $y=1+3x^2$, $x=0$, $y=0$, $x=2$

9 Find the $y$ coordinate of the c g of the uniform lamina bounded by $y=1/x^2$, $y=0$, $x=\frac{1}{3}$, $x=1$

10 Find the coordinates of the c g of the uniform lamina bounded by $y=1/x^3$, $x=\frac{1}{2}$, $x=1$, $y=0$

11 Show that the $x$ coordinate of the c g of the uniform lamina bounded by the $x$ axis and $y=ax^2(1-x)$ is 0·6 for all positive values of $a$ If $a=7$ show that the $y$ coordinate of the c g is 0 4

## 5.17. Average velocity from the velocity-time formula

The velocity of a train $t$ min after leaving a station is $v$ ft per min, where $v=30t^2(10-t)$ After 10 min the train stops at the next station The average velocity between the stations is the constant velocity at which the train would travel the distance between the stations in the same time

$$\text{This distance} = \int_0^{10} 30t^2(10-t)\,dt$$
$$= \int_0^{10} (300t^2-30t^3)\,dt$$
$$= \left[100t^3-\tfrac{30}{4}t^4\right]_0^{10}$$
$$= 10^5-\tfrac{3}{4}10^5 = 25000 \text{ ft.}$$

Hence the average velocity is $\dfrac{25000}{10}$ ft. per min.

$$= 2500 \text{ ft. per min.}$$
$$\doteqdot 28{\cdot}4 \text{ m.p.h.}$$

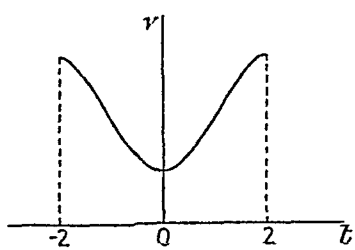
### Exercise 5.Q

1. A body moves on the $x$-axis so that at $t$ sec. its velocity, $v$ cm. per sec., is given by $v = 2t+t^2$. Find the average velocity from $t = 3$ to $t = 6$.

2. The speed of a train for the first 3 min. after leaving a station is $120(t+1)^2$ yd. per min., where $t$ is the time in min. from the start. Find the average velocity during these 3 min.

3. A train is checked by signals so that its velocity $v$ yd. per min. at $t$ min. is given by $v = 500+600t^2-75t^4$.

The check begins at $t = -2$ when the brakes are applied. Before the train is brought to rest, the line is cleared and the train regains full speed at $t = +2$. Find the average speed of the train from $t = -2$ to $t = +2$.




4. The tide flows past a lightship for 6 hr. in each direction. The velocity, $v$ m.p.h., in one direction, is given at $t$ hr. after it begins to flow in that direction by the formula $v = \frac{1}{27}t^2(6-t)^2$. Find the average velocity for the 6 hours. Find also the maximum velocity during this time.

5. If a body moves with constant acceleration so that its velocity is given by
$$v = a+bt,$$
where $a, b$ are constants, show that the average velocity from $t = 0$ to $t = x$ is the velocity of the body at $t = x/2$.

6. If the body moves so that $v = a+bt^2$, where $a, b$ are constants, show that the average velocity from $t = 0$ to $t = x$ is the velocity of the body when $t = x/\sqrt{3}$.

## 5 18 Mean values

In general if the velocity time formula for the motion of a body is known, its average velocity from $t = a$ to $t = b$ is

$$\frac{1}{b-a}\int_a^b v\,dt$$

This definition may now be carried over to any function of $x$ If $y$ is a function of $x$, then the *mean value* of $y$ for values of $x$ from $x = a$ to $x = b$ is

$$\frac{1}{b-a}\int_a^b y\,dx$$

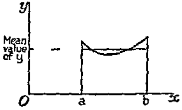



Fig 5 39

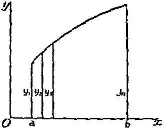



Fig 5 40

Since $\int_a^b y\,dx$ is the area under the graph of the function from $x = a$ to $x = b$, the mean value of $y$ is the height of a rectangle whose base is $(b-a)$ and whose area equals the area under the curve from $x = a$ to $x = b$

It is interesting to compare this definition with that for finding the mean of a set of numbers

To find a batsman's average for the season (i e his mean score for the season) we add all the scores together and divide by their number With a continuously varying quantity like $y$ we cannot do this, but the definition given above is a generalization of the arithmetical one

Fig 5 40 represents the part of a graph between $x = a$ and $x = b$ Take $n$ equidistant values of $y$ and represent them by $y_1, y_2\ y_3 \quad y_n$ The arithmetic mean of these values is

$$\frac{y_1+y_2+y_3+\quad+y_n}{n} = \frac{y_1+y_2+y_3+\quad+y_n}{b-a}\times\frac{b-a}{n}$$

Now $(b-a)/n$ is the change of $x$ between two successive values

of $y$. We therefore write $(b-a)/n = \delta x$. Then the mean is

$$\frac{(y_1+y_2+y_3+\ldots+y_n)\,\delta x}{b-a} = \frac{y_1\,\delta x+y_2\,\delta x+y_3\,\delta x\ldots+y_n\,\delta x}{b-a}.$$

Now let $n$ increase beyond any number, however large, so that $\delta x \to 0$. Then, by the definition of the integral,

$$\text{the mean tends to} \quad \frac{1}{b-a}\int_a^b y\,dx.$$

### Exercise 5.R

1–5. Find the mean value of $y$ for the given range of values of $x$.

1. $y = x^2$, $x = 2$ to $x = 4$.
2. $y = 9-x^2$, $x = -3$ to $x = 3$.
3. $y = x^3$, $x = 0$ to $x = 4$.
4. $y = 1/x^2$, $x = 1$ to $x = 10$.
5. $y = 4+4x-x^2-x^3$, $x = -1$ to $x = 2$.

6. Show that, whatever the value of $a$, the mean value of $y = a^2-x^2$ from $x = -a$ to $x = a$ is $\frac{2}{3}$ of the maximum value of $y$.

7. During the passage of an anticyclone, the height of the barometer $y$ in. is given in terms of the time, $x$ days, by the formula

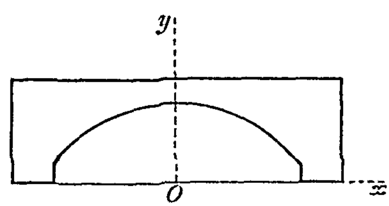
$$y = 29+\frac{3x}{2}-\frac{x^2}{4}.$$

Find the mean height during the four days $x = 0$ to $x = 4$.

If $x = 0$ is 10 a.m. on the first day, find the average of the five 10 a.m. readings at $x = 0, 1, 2, 3, 4$.

8. The height, $y$ ft. above the river, of the arch of a bridge is given in terms of distance, $x$ ft., from the centre line by

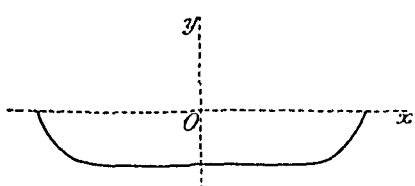
$$y = 8-(x^2/24).$$




The span of the bridge is from $x = -12$ to $x = +12$. Find the mean height of the arch. What is the greatest height of the arch?

9. Find the mean value of the function $y = \frac{1}{3}(x^8-1)$ from $x = -1$ to $x = 1$.

If this curve is the cross-section of the bottom of a canal on the scale unit $x$ and $y = 10$ yd., give, in feet, (1) the maximum depth, (2) the mean depth of the canal.



10 A barrel is obtained by the rotation of the curve $y = 1 + \frac{1}{2}x - \frac{1}{8}x^2$ about the $x$ axis from $x = 0$ to $x = 4$ (unit $x$ and $y = 1$ ft) Find the mean diameter of the barrel

Iron hoops are put round the barrel at $x = 0, 1, 3, 4$ Find the average of the diameters of these hoops

11 On a summer afternoon at $t$ hr after noon the temperature, $T°$ F, is given by

$$T = 70 + 2t - \frac{t^3}{25}$$

Find the mean temperature from noon to 5 p.m

## 5.19. Approximate integration

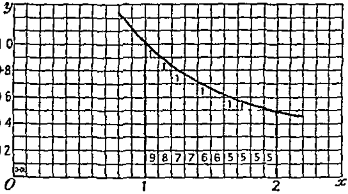



FIG 5 41

Since $\int_a^b y\,dx$ is the area under the graph of $y$ from $x = a$ to $x = b$, we can find an approximate value of the integral by drawing the graph on squared paper and counting squares This method is useful in two cases

**19.1.** We may be unable to integrate in the usual way because $y$ is a function of $x$ which we have not met as a gradient For example, $1/x$ has not appeared as the gradient of any of the functions considered so far If we wish to calculate $\int_1^2 1/x\,dx$, we are unable to do so exactly, at present, but we can find its value approximately by drawing the graph of $y = 1/x$ from $x = 1$ to $x = 2$ and estimating the area under this portion of the graph by counting squares Fig 5 41 shows the graph from $x = 1$ to $x = 2$ and the method of organizing the count of squares The number of complete squares in each column is written in the

column and the part squares at the tops of the columns are counted separately. It would take too long to estimate each part square as a fraction of a square, but it can quickly be estimated which part squares are less than a half square and these are ignored in the count. The remaining part squares, which are greater than, or equal to, a half square, are counted as 1 square. In Fig. 5.37 these squares are marked 1. This estimation of the area of the part squares constitutes the approximation of this method of evaluating an integral. Clearly, the result is likely to be more accurate (and the work more tedious) when the squares are small or the scale is large.

In the present case the number of complete squares is

$$9+8+2\times 7+2\times 6+4\times 5 = 63.$$

The part squares are estimated as equivalent to 6 squares.

Hence the area under the graph is estimated as 69 squares.

Now the side of a square parallel to the $x$-axis represents 0·1 on the $x$ scale and the side parallel to the $y$-axis represents 0·1 on the $y$ scale.

Hence each square represents 0·01, and the value of the area under the graph is $69\times 0{\cdot}01 = 0{\cdot}69$.

Hence $$\int_1^2 \frac{dx}{x} \doteqdot 0{\cdot}69.$$

EXERCISE 5.S

1–3. Evaluate, approximately, the integrals below. The answers to the questions are given to the accuracy obtainable on $\frac{1}{4}$-in. squared paper with the scales suggested.

1. $\int_2^3 \frac{dx}{x}$; unit $x = 2\frac{1}{2}$ in., unit $y = 5$ in.

2. $\int_0^4 \sqrt{x}\, dx$; unit $x = \frac{1}{2}$ in., unit $y = 1\frac{1}{4}$ in.

3. $\int_0^3 2^x\, dx$; unit $x = \frac{1}{2}$ in., unit $y = \frac{1}{4}$ in.

It is as well to have some idea of the accuracy of the method.

4. Evaluate $\int_1^3 x^2\, dx$ (*a*) approximately, taking unit $x = \frac{1}{2}$ in., unit $y = \frac{1}{4}$ in., (*b*) exactly.

**19 2.** The second case in which this approximate method of evaluating integrals is useful is when $y$ is not defined in terms of $x$ by a formula but by a table of values For example, in the following table we are given corresponding dimensions of a flask of which we wish to find the volume

| Distance along the axis from the base in inches | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|
| Radius of section perpendicular to axis in inches | 3 | 3 7 | 4 1 | 4 2 | 4 | 3 3 | 2 | 1 | 0 8 | 0 8 |


Fig 5 42

By drawing a curve through the points plotted from this table we draw a half section of the flask (Fig 5 42) If we take the axis of symmetry as the $x$ axis and a radius of the base as the $y$ axis, this curve generates the surface of the flask as it rotates about the $x$ axis

Consider a rectangle $y\,\delta x$ The volume generated by the rotation of this rectangle is $\pi y^2\,\delta x$ and so the volume of the flask is $\int_0^{18} \pi y^2\,dx$ If $z = \pi y^2$, this integral becomes $\int_0^{18} z\,dx$, and is recognized as the area from $x = 0$ to $x = 18$ under the graph of $z$ plotted against $x$ We now draw this graph from a table of values of $z = \pi y^2$ when $x$ has the values 0, 2, 4, , 18

| $z$ (sq in ) | 28 | 43 | 53 | 55 | 50 | 34 | 13 | 3 | 2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ (in ) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |

We draw a curve through the plotted points and estimate the area under the graph by counting squares (Fig 5 43) There are 90 complete squares and 18 part squares† to be included in the count, and so the estimated area under the curve is 108 squares

† The rule for including part squares should be used with discretion For example from $x = 14$ to $x = 18$ the graph is approximately parallel to the $x$ axis and there is no chance of compensation As there are 4 squares of which about $\frac{1}{4}$ should be included in the count we mark one square although it is less than half a square

The side of a square parallel to the $y$-axis represents a cross-sectional area of 5 sq. in., and the side parallel to the $x$-axis represents 1 in. Hence each square represents a volume of 5 cu. in. The volume of the flask $\doteqdot 108 \times 5 = 540$ cu. in.

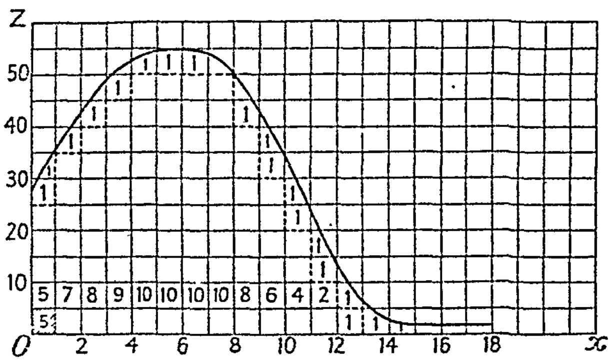


FIG. 5.43

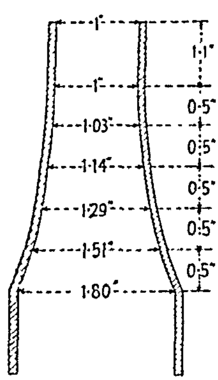
## EXERCISE 5.T

1. A vase, 10 in. high, is a volume of revolution and the radius of its section at various distances from the base is given in the table:

| | | | | | | |
|---|---|---|---|---|---|---|
| Radius (inches) . . . . | 2 | 2·9 | 3·2 | 3 | 2·5 | 1·8 |
| Distance from base (inches) . | 0 | 2 | 4 | 6 | 8 | 10 |

Sketch the vase and find its volume.

2. The figure shows the cross-section of a 1-in. nozzle which is fitted to the end of a fire-hose. From the dimensions on the sketch we obtain the following table:



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Distance from orifice (inches) . . | 0 | 1·1 | 1·6 | 2·1 | 2·6 | 3·1 | 3·6 |
| Area of cross-section (sq. inches) . | 0·78 | 0·78 | 0·82 | 1·02 | 1·31 | 1·79 | 2·54 |

Draw the curve of areas and find the volume of this portion of the nozzle, approximately.

3. A mound of earth 3 ft. high in the centre is in the form of a solid of revolution with a vertical axis. The radius of the mound at various heights is given in the table. Estimate the volume of the mound.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Height (ft.) . . . . | 0 | 0·5 | 1·0 | 1·5 | 2·0 | 2·5 | 3·0 |
| Radius (ft.) . . . . | 6·0 | 4·5 | 2·9 | 1·9 | 1·1 | 0·6 | 0 |

4 On a map, contours are drawn to show heights of 300, 250, 200, 150, 100 ft The top of a hill is 300 ft high and the flat ground round its base is 100 ft high Find the volume of the hill in cu yd to 2 sig fig if the areas enclosed by the contours are estimated from the map to be

| Area enclosed (sq yd) | 0 | 950 | 1700 | 3000 | 5500 |
|---|---|---|---|---|---|
| Contour | 300 | 250 | 200 | 150 | 100 |

5 The density of the air at various heights is determined on a certain day and found to be

| Height (ft) | 0 | 1000 | 2000 | 3000 | 4000 | 5000 |
|---|---|---|---|---|---|---|
| Density (oz per cu ft) | 1 27 | 1 19 | 1 20 | 1 14 | 1 10 | 1 07 |

Find the weight, to the nearest 10 lb, of a column of air 5,000 ft high from the surface and 1 sq ft cross section

6 By considering the area of the circle $x^2+y^2 = a^2$ in the positive quadrant, show that the exact value of $\int_0^a \sqrt{(a^2-x^2)}\, dx$ is $\frac{1}{4}\pi a^2$

What is the value of $\int_{-a}^{a} \sqrt{(a^2-x^2)}\, dx$?

7 The speed of a train leaving a terminus is given at various times after the start in the following table

| Speed (m p h) | 0 | $5\frac{1}{2}$ | $9\frac{1}{2}$ | $15\frac{1}{2}$ | $19\frac{1}{2}$ | $21\frac{1}{2}$ | $25\frac{1}{2}$ | 34 | 44 | 52 |
|---|---|---|---|---|---|---|---|---|---|---|
| Time from start (min) | 0 | $\frac{1}{2}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Find the distance travelled during the 8 min and the average speed during that time

8 The speeds of an electric train at various times after leaving one station until it stops at the next station are

| Speed (m p h) | 0 | 13 | 33 | $39\frac{1}{2}$ | 40 | 40 | 36 | 15 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| Time (min.) | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 | $3\frac{1}{4}$ | $3\frac{1}{2}$ |

Find the distance between the stations and the average speed of the train

## 5.20. Approximating to the area under a curve

The method of counting squares is often laborious, and if great accuracy is not expected from the result the labour involved may not be justified For example, in Ex. 5 T, No 5, the air density is only given at intervals of 1000 ft, so that the curve under which the area is calculated is not itself very accurate For such cases, a quicker method consists in approximating to the curved boundary by a series of straight lines

Consider the area bounded by the $x$- and $y$-axes and a curve passing through the points

| $x$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $y$ | 3 | 3·9 | 2·6 | 0 |

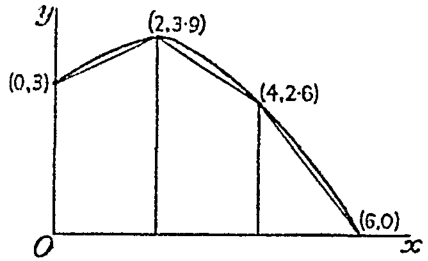



Fig. 5.44

We plot the four points and draw a curve through them. To estimate the area under this curve we replace the curved boundary by the straight lines joining (0, 3), (2, 3·9); (2, 3·9), (4, 2·6); (4, 2·6), (6, 0), and calculate the areas of the three polygons so formed, the first two being trapeziums and the third a triangle.

Area of first trapezium $= \frac{2}{2}(3+3{\cdot}9) = 6{\cdot}9$.
Area of second trapezium $= \frac{2}{2}(3{\cdot}9+2{\cdot}6) = 6.5$.
The area of the triangle is $\frac{2}{2}\times 2{\cdot}6 = 2{\cdot}6$.

Hence the estimated total area is 16. We can now see another advantage of this method; it can be performed without drawing the graph at all. The disadvantage is that, especially in a case where the boundary bends the same way for the whole length, the approximation is apt to be rather rough. In the present example each polygon has a smaller area than that under the corresponding portion of the curve and so the errors add together. By counting squares, I estimate the area under the curve to be 19.

## Exercise 5.U

1. Find approximately the area bounded by a smooth curve through the points (0, 0), (1, 1), (2, 1·4), (3, 1·7), and the lines $x = 3$, $y = 0$.

2. Find approximately the area bounded by a smooth curve through (4, 2), (6, 1·7), (8, 1·2), (10, 1), and the lines $x = 4$, $x = 10$, and $y = 0$.

3. The speed of a train at various times from the start is given in the following table:

| | | | | | |
|---|---|---|---|---|---|
| Speed (m.p.h.) . | 0 | 12 | 28 | 38 | 45 |
| Time (min.) . | 0 | 2 | 4 | 6 | 8 |

Find the distance run in the 8 min. (This question is based on the test run shown in the frontispiece.) You can check your answer by noting the distance run when the speed of 45 m.p.h. is attained.

4 The density of the air at various heights up to 2 km is greater in winter than in summer, on the average, by the following amounts

| | | | | | |
|---|---|---|---|---|---|
| Height above sea level (km) | 0 | 0.5 | 1 | 1.5 | 2.0 |
| Excess density in winter over that in summer (gm per cu metre) | 63.2 | 53.1 | 47.6 | 37.7 | 29.8 |

Estimate, in kg to the nearest unit, the increased weight in winter of a column of air 2 km high and 1 sq metre in cross section


5 The sketch shows a section of an aeroplane wing, the under surface being flat $A$ is the leading edge and the section may be drawn from the following table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Distance from leading edge, $x$ (ft) | 0 | 0.25 | 0.5 | 1 | 2 | 4 | 6 | 8 | 10 |
| Height above $AB$, $y$ (ft) | 0.11 | 0.44 | 0.61 | 0.81 | 0.99 | 1.02 | 0.90 | 0.59 | 0.08 |

Estimate the area of the section

## Miscellaneous Exercise 5 X

1. Evaluate $\int_1^3 (3x-2)\,dx$

2 Evaluate $\int_{-1}^{5} (2x^2-2x+3)\,dx$.

3 Find the area under the curve $y = 4x^2+2$ from $x = 0$ to $x = 3$ Show that the curve $y = x^3+\frac{1}{4}$ bisects this area

4 Find the volume obtained by rotating the part of the curve $y = x^3$ between (1, 1) and (2, 8) about the $x$ axis

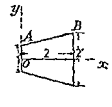
5 Find the volume of the bowl formed by revolving $4y = x^2$ about the $y$ axis from $y = 9$ to $y = 16$ (units inches)

6 Evaluate $\int_{\frac{1}{2}}^{1\frac{1}{2}} (6x^2+4/x^2)\,dx$

7 Find the area bounded by $y = 2+3x-2x^2$ and the $x$ axis

8 Draw a sketch of the areas represented by $\int_1^2 dx/x$, $\int_2^4 dx/x$

Show approximately, by counting squares, that these areas are equal




9 Find the equation of the straight line $AB$ shown in the sketch Find the volume of the bucket generated by the rotation of $AB$ about the $x$ axis (unit $x$ and $y$ = 1 ft)

10. Find the c.g. of the uniform solid formed by the rotation of the part of the curve $y = x^3$ from (0, 0) to (1, 1) about the $x$-axis.

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11. Evaluate $\int_1^2 \frac{(x-2)^2}{x^4}\,dx$.

12. Find the area bounded by $y = 6x^3+6x^2-12x$, $x = -2$, $x = -1$, $y = 0$.

13. Find the volume obtained by rotating the part of the curve $y^2 = 2x+4$ between (0, 2) and (6, 4) about the $x$-axis.

14. The areas from $x = 0$ to $x = h$ under (1) $y = x^2$, (2) $y = x^3$ are equal. Find $h$.

15. The differential equation of a family of curves is $dy/dx = 1+3x^2$. Find the equation of the curve such that the area under it from $x = 0$ to $x = 1$ is $2\frac{3}{4}$.

16. Find the c.g. of the area bounded by $y = 1+x^2$, $y = 0$, $x = 0$, $x = 1$.

17. Find the area bounded by $y = 8x-3x^2$ and $y = 2x$.

18. $y^2 = 4x$ is rotated about the $x$-axis from $x = 0$ to $x = 4$. Find $h$ if the volume generated from 0 to $h$ equals the volume generated from $h$ to 4.

19. Evaluate $\int_0^3 \{1+4(2+x)^2\}\,dx$.

20. A canal is to be excavated and the cross-section is to be a rectangle of depth 10 ft. and width 30 ft. The cost, $c$ shillings, of excavating a cubic yard at depth $x$ ft. is given by $c = \frac{1}{2}+x/20$. Find the cost of excavation per yard length.

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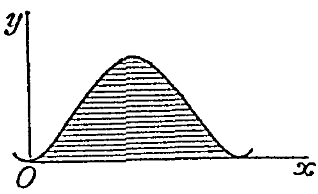
21. Evaluate $\int_{\frac{1}{2}}^{1} \left(\frac{1}{x^2}-1\right)\left(\frac{3}{x^2}+1\right)dx$.




22. The sketch shows part of the curve $y = x^2(x-2)^2$. Find the shaded area.

23. A bobbin is formed by the revolution of $4y = x^2$ about $y+1 = 0$ between $x = -2$ and $x = +2$ (units: inches). Find its volume.

24. Show that the mean height of $y = x^3+8$ from $x = -2$ to $x = 2$ equals the intercept made by the curve on the $y$-axis and explain the result geometrically, in terms of areas.

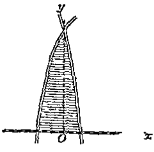
25. Find the area bounded by $y = 5$, $5y = 4x^2$.

26. The velocity-time formula for the run of a train during two hours is $v = 60(1-t^2)$ from $t = -1$ to $t = +1$ (units: miles, hours). Find (a) the maximum speed, (b) the average speed.

27 Find the volume obtained by rotating $16y = x^2$ from $(0, 0)$ to $(4, 1)$ (a) about the $y$ axis, (b) about the $x$ axis

28 Find the c g of the uniform solid formed by the rotation of $y^2 = 3x$ from $(0, 0)$ to $(3, 3)$ about the $x$ axis

29 A field is bounded by a straight railway line ($y = 0$) and two roads whose equations are $y = 12+x-x^2$, $y = 12-4x-x^2$ (unit $x$ and $y = 1$ chain) The corners of the field are $A \equiv (-3, 0)$, $B \equiv (0, 12)$, $C \equiv (2, 0)$ Find which equation applies to $AB$ and which to $BC$, and determine the area of the field in square chains




30 Draw the graph of $y = 1/(1+2x)$ from $x = 0$ to $x = 1$, taking unit $x$ and $y$ to be $2\frac{1}{2}$ in ($\frac{1}{4}$ in paper) Find an approximate value for $\int_0^1 \frac{dx}{1+2x}$

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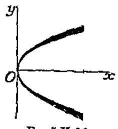
31 Find the smaller volume cut off a sphere of radius 2 ft by a plane whose perpendicular distance from the centre is 1 ft

32 Find the c g of the area bounded by $y = 6(1-x^2)$ and $y = 0$

33 Find the area bounded by $y = 3x^2$ and $y+3x = 6$

34 The velocity of a body moving in a straight line is $3t^2/10$ cm per sec after $t$ sec In what distance does its velocity increase from 30 cm per sec to 120 cm per sec ?

35 Find the value of $c$ if the straight line $3y = x+c$ bisects the area bounded by $y = 1+x^2/3$ $x = 3$ $y = 0, x = 0$




Ex 5 X 36

36 Find the volume of the shell bounded by the surfaces generated by the rotation about the $x$ axis of $y^2 = 10x$ and $y^2 = 11x$ from $x = 0$ to $x = 1$

37 Find the volume generated by the rotation of $y = 2x^2$ about $y = 2$

38 (a) Find the mean value of the squares of all the numbers from 10 to 20 inclusive, (b) Find the mean value of the squares of the integers from 10 to 20 inclusive

39 The mass per unit area of a rectangular plate 2 ft wide and 3 ft long is constant across its width, but at a distance $x$ ft from one 2 ft edge it is $(15+2x)$ lb per sq ft Find the mass of the plate and the distance of its c g from the 2 ft edge

40 Show that the area under $y = ax^2+b$ from $x = -h$ to $x = h$ is $\frac{2}{3}h(y_1+2y_2)$ where $y_1$ is the height of the area at either side and $y_2$ is its height at the middle

## Miscellaneous Exercise 5.Y

1. Find the area between the curve $y = 3x(2-x)$ and the $x$-axis.

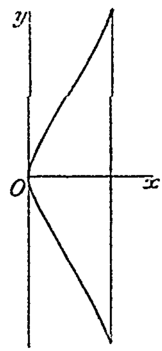
2. Find the area between $y = x^3$ and $y = 4x$ in the first quadrant.

3. Find the area bounded by $y = x^2$ and $y = 16$. Show that the line $y = 4$ divides this area into two parts whose areas are in the ratio 1:7, the smaller part being nearer the origin.

4. Show that if $y = 12x-18x^2+12x^3-3x^4$, $2\int_0^1 y\,dx = \int_0^2 y\,dx$.

5. Show that $\int_{\frac{1}{2}}^{2} (x^2+\frac{1}{4})\,dx = \int_{\frac{1}{2}}^{2} 2/x^2\,dx$. Show what the result means on a sketch of the curves $y = x^2+\frac{1}{4}$, $y = 2/x^2$.

6. Find the volume generated by the revolution, about the $x$-axis, of $y^2 = 4x+x^4/2$ from $x = 0$ to $x = 2$.



Ex. 5.Y 6

7. Show that the ratio of the volume obtained by rotating the part of the curve $y = 1+x^4$ between $x = 0$ and $x = 1$ about the $x$-axis to the volume obtained by rotating the part of the curve $y = 1+x^2$ from $x = 0$ to $x = 1$ about the same axis is 17/21.

8. A cap of a sphere of radius 10 in. is cut off by a plane 8 in. from the centre. Find the volume cut off.

9. A wooden post 8 ft. long tapers so that at distance $x$ ft. from its thick end its weight per unit length is $(15-3x/4)$ oz. per ft. Find (i) weight of post; (ii) distance of its c.g. from the thick end.

10. The shape of a church steeple is given by the rotation of $y = 4/x$ about the $y$-axis from $y = 4$ to $y = 16$, unit $x$ and $y$ being 10 ft. Find the volume of the steeple.

11. The pressure of water at depth $h$ ft. below the surface is $60h$ lb. wt. per sq. ft. Find the pressure on a vertical lock-gate whose immersed area is a rectangle 10 ft. wide and 16 ft. deep.

12. Find the volume obtained by revolving $y = 2x^3$ from (0, 0) to (1, 2) about the line $y = 3$.

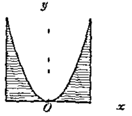
13. Find the volume generated by the revolution of $y = 4x^2$ from (0, 0) to (1, 4) (*a*) about the $y$-axis, (*b*) about the line $y+3 = 0$.

14. Show that $\int_0^1 3x^2(1-x)^2\,dx = \int_0^1 2x^3(1-x)\,dx$.

15. Show that $\int_{\frac{1}{2}}^{2} \frac{1+x^2}{x^4}\,dx = \int_{\frac{1}{2}}^{3} (1+x^2)\,dx$.

16. Find the area bounded by the part of the curve $y = x^2+x^3$ from $(-1, 0)$ to $(2, 12)$ and the straight lines $y = 12$, $x+1 = 0$.

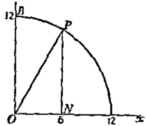
17 A 'tiddly winks' cup is made from a cylindrical block of wood of radius $a$ and height $2a$ by excavating the interior until the inner boundary of the cup is given by rotating $y = 2x^2/a$ about the $y$ axis which coincides with the axis of the cylinder Show that the completed cup weighs half the original cylinder [Ignore the fact that the cup has no thickness at $O$ It would not be possible to make such a cup, but the result would be approximately true for an actual cup with a thin base ]




18 A symmetrical pyramid has a square base of side $a$ and height $h$ Prove that its volume is $\frac{1}{3}a^2h$ If it is uniform, find the distance of its c g from the vertex

19 A body moves in a straight line so that its velocity $v$ f p s at $t$ sec after the start is given by $v = 12t - 3t^3$ Find the distance travelled from the start when the acceleration is instantaneously zero Is the velocity a maximum or minimum then? Find also the accelerations at the two instants when the velocity is zero

20 *To find an approximate value for* $\pi$ Take the quadrant of the circle $x^2+y^2 = 144$ for which $x$ and $y$ are both positive Calculate $y$ when $x = 0, 2, 4, 6, 8, 10, 12,$ and, replacing the circular boundary between each pair of consecutive points by its chord, find approximately the area of the quadrant Equate this to $\frac{1}{4}\pi \times 12^2$ and find an approximate value of $\pi$ (to 2 sig fig )

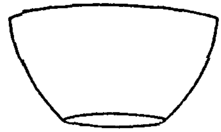



A better approximation may be found as follows Find the area under the circle from $x = 0$ to $x = 6$, calculating $y$ for $x = 0, 1, 2, 3, 4, 5, 6$ and working to 4 sig fig Take away the area of the triangle $OPN$ and obtain the area of the sector $BOP$ What is the cosine of angle $PON$? What is angle $BOP$? What fraction of the complete circle is the area of the sector? Hence, find an approximation to $\pi$ to 3 sig fig

21 Find, approximately, the value of $\int_1^2 \log_{10} x \, dx$ [If the graph, $y = \log x$, is drawn with unit $x = 2\frac{1}{2}$ in and unit $y = 5$ in, the result should be correct to two decimal places ]

22 A block of stone is made in the shape of a solid of revolution, the base being a circle of radius 1 ft The height of the stone is 1 ft and its diameter at various heights above the base is given in the following table

| Diameter (ft.) | 2 | 1·94 | 1·74 | 1·42 | 1 |
|---|---|---|---|---|---|
| Height (ft.) | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |

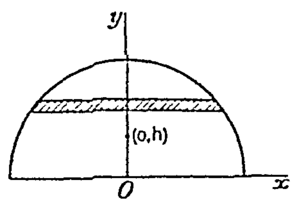


Find, approximately, the volume of the stone.

23. Find, to two decimal places, the value of $\int_0^1 x\sqrt{(1-x^2)}\,dx$ by drawing the graph of $y = x\sqrt{(1-x^2)}$, taking unit $x$ and $y$ = 5 in. and counting squares. To save your time, a table of values is given, but some of the entries should be verified.

| $x$ | 0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 0·95 | 1·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x\sqrt{(1-x^2)}$ | 0 | 0·10 | 0·20 | 0·29 | 0·37 | 0·43 | 0·48 | 0·50 | 0·48 | 0·39 | 0·30 | 0 |

If $(0, h)$ is the c.g. of the semicircle with radius 1 shown in the sketch, prove that




$$\frac{\pi h}{2} = \int_0^1 2y\sqrt{(1-y^2)}\,dy$$

and deduce that $h \doteqdot 4/3\pi$.

Actually this formula is exact, but this can only be proved when the integral can be evaluated exactly.

24. A solid is formed by rotating $y^2 = 4x$ from $(0, 0)$ to $(4, 4)$ about the $x$-axis. Find the distance of its c.g. from the origin.

25. Find the area between $y = 5(1-x^2/4)$ and the $x$-axis. Find the coordinates of its c.g.

26. Find the coordinates of the c.g. of the area in the first quadrant bounded by $y = 9-x^2$, $x = 0$, $y = 0$.

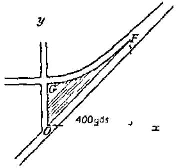
27. An inn sign 3 ft. by 4 ft. hangs vertically at right angles to the wind. When the velocity of the wind is $v$ m.p.h. the pressure exerted on the sign is $0{\cdot}003v^2$ lb. wt. per sq. ft. The wind is blowing at 10 m.p.h. when a squall begins and during the next 2 min. its velocity is given by $v = 10(1+t/2)$, where $t$ is the time in min. Find the mean pressure of the wind on the sign during the 2 min.

28. Verify that $(h, h^2)$ is a point on the curve $y = x^2$.

(i) Find the ratio of the volume generated by rotating the part of the curve $y = x^2$ between $(0, 0)$ and $(h, h^2)$ about the $x$-axis to that obtained by rotating it about the $y$-axis.

(ii) Find $h$ if these volumes are equal.

29. The acceleration of a body is $(5+4t)$ f.p.s.$^2$ and its velocity at $t = 0$ is 10 f.p.s. Find the distance travelled in the fourth second (from $t = 3$ to $t = 4$).




30 The sketch shows the plan of a village green $OFG$ and the roads running by it

The equation of $GF$ is $8y = 16+x^2$, where the unit of $x$ and $y$ is 100 yd, $OF$ touches $GF$ at the fork, $F\ GOx$ is a right angle Verify that the tangent at $F$ passes through the origin as shown in the sketch Find the area of the green to the nearest acre

31 A vase is formed by the rotation of the curve $y = 6/x + x/6$ about the $x$ axis from $x = 2$ to $x = 18$, the units of $x$ and $y$ being inches Show that the diameter of the base ($x = 2$) equals the diameter of the top ($x = 18$) and give their value What is the least diameter of the vase? Find its volume

32 Find the area enclosed by $y = 11+7x-3x^2$ and $y = x+2$

33 Find the area enclosed by the curves $y = 4-x^2$ and $y = 3x^2$ Find the volume swept out by this area if it is rotated about the $y$ axis through two right angles

34 Find the area bounded by $y = 1+x^3$, $y = 0$, $x = 0$, $x = 2$ Find the coordinates of the c g of this area Find the volume of the solid obtained by revolving the area about $y = 0$ (Use one of the integrals already evaluated)

35 Find the volume generated by the rotation of $y = 1/x^2$ from $(1, 1)$ to $(2, \frac{1}{4})$ about $y = 1$

## Miscellaneous Exercise 5 Z

1 Find the area between $y = 3(1-x)(x-5)$ and the $x$ axis

2 Show that (i) $\int_0^2 (3x^2+2)\,dx + \int_2^5 (3x^2+2)\,dx = \int_0^5 (3x^2+2)\,dx$,

(ii) $\int_{-a}^{a} (3x^2+2)\,dx = 2\int_0^a (3x^2+2)\,dx$, and illustrate the results by considering areas under the curve $y = 3x^2+2$

3 Show that $\int_1^4 \left(\frac{x^2}{2} - \frac{1}{2x^2}\right) dx = \int_1^2 \left(u^5 - \frac{1}{u^3}\right) du$

4 Find the area enclosed by $y = kx^2$ and $y = ax$ If $a = 3$ and $k$ is increased from 2 to 2 1, find the approximate change in the area

5 A spherical loaf of radius 5 in is cut into 10 slices each 1 in thick Find the ratio of the volumes of the first slice (the crust) and the third slice

6. (*a*) A lens is formed by rotating the curves $y^2 = 8x+9$ and $y^2 = 9(1-x)$ from $y = 0$ to $y = 3$ about the $x$-axis. Find its volume.

(*b*) More generally, show that the curves

$$by^2 = h^2(x+b) \text{ and } ay^2 = h^2(a-x)$$

meet the $x$-axis at $B \equiv (-b, 0)$ and $A \equiv (a, 0)$ respectively, and that both curves meet the $y$-axis at $C \equiv (0, h)$, $D \equiv (0, -h)$.

Show further that the volume generated by the rotation of $BCA$ about the $x$-axis is

$$\frac{\pi h^2}{2}(a+b).$$

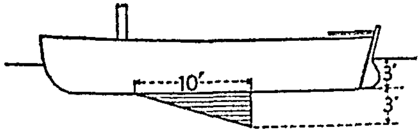
7. Find the equation of the normal to $4y = x^2$ at $(2, 1)$. Find the volume generated by revolving the area bounded by the curve, the normal and the $x$-axis about this axis.

8. The portion of the curve $y = a^2-x^2$ for which $y$ is positive is rotated (*a*) about the $x$-axis, (*b*) about the line $y = 2a^2$. Show that the ratio of the volumes generated does not depend upon $a$ and find its value.

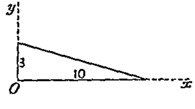
9. The area bounded by $y = 2+2x^2$, $x = 0$, $y = 0$ and the variable boundary $x = x$ is $A = \int_0^x (2+2x^2)\,dx$. What is the value of $dA/dx$ when $x = 4$? If $x$ increases from 4 to 4·2, find approximately the increase in the area.

10. A bowl is obtained by revolving $3y = x^2-1$ about the $y$-axis from $y = 0$ to $y = 1$ (the units are feet), and water is poured into the bowl. Find the volume of water when the surface is $h$ ft. above the bottom of the bowl. When the depth is 8 in., find approximately the change of depth if an extra 0·1 cu. ft. of water is added.

11. A bowl has the shape of a solid of revolution whose axis is the $y$-axis. The bottom and top of the bowl are at $y = 0$ and $y = 2$. When the depth of water in the bowl is $h$, its volume is $\pi(h^3+4h)$. Find the radius of the bowl at height $h$ above the base. What is the equation of the boundary curve?




12. The centre-board of a yacht, when lowered, is a right-angled triangle with a horizontal side of 10 ft. and a vertical side of 3 ft. The horizontal side is 3 ft. below the surface of the water. If the pressure of the water at depth $h$ is $60h$ lb. wt. per sq. ft., find the pressure of the water on one side of the centre-board.




13 Placing the centre board of No 12 in relation to the axes as shown in the sketch, find the position of its c g

14 *Calculations in yacht designing*

14 1 A designer draws the lines of a yacht on a scale of $\frac{1}{4}$ in $= 1$ ft He finds that, on the drawing, the area of the under water cross section of the design at various distances from the bow is as follows

| Distance from bow (in ) | 1 | 2 | 3 | 4 | 5 | 6 | 6 5 |
|---|---|---|---|---|---|---|---|
| Area of cross-section (sq in ) | 0 | 0 38 | 0 82 | 0·99 | 0 70 | 0 18 | 0 |

Represent these figures on a graph and estimate, as accurately as you can by counting squares, the volume of the yacht This is called the displacement and when divided by 35 it gives the tonnage

14.2 Here are some figures from an actual design † The yacht is 16 ft long and the under water cross sectional area is given at 10 equally spaced sections, which are, therefore, 1 6 ft apart

| Number of the section (from bow to stern) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Area of section (sq ft ) | 0 00 | 0 80 | 2 50 | 4 34 | 6 00 | 7 20 | 7 36 | 6 05 | 3 96 | 1 66 | 0·76 |

Draw a graph of these areas This graph is called the curve of areas The designer makes the volume of the under water body of the yacht 64 32 cu ft Check this result

14 3 Frequently the yacht designer finds the area under the curve of areas by the method of section 5 20 If the areas of the sections 0, 1, 2, , 9, 10 are $y_0$ $y_1, y_2,$ $, y_9, y_{10}$ sq ft and the sections are $h$ ft apart, show that the calculation of the area of the curve of sections by the method of section 5 20 leads to the following approximation to the volume of the yacht

$$V = \left(\frac{y_0}{2}+y_1+y_2+ \quad +y_9+\frac{y_{10}}{2}\right)h \text{ cu ft}$$

This is called the *trapezoidal rule*

Calculate the under water volume of *Paida* in this way

14 4 Show that if $nh = (b-a)$ and $y_0, y_1, y_2, \ \ ,y_n$ are the values of $y$ at $x = a, a+h, a+2h, \ \ ,b,$

† *Paida* from *Cruising Yachts Design and Performance* by T Harrison Butler (Robert Ross & Co Ltd.)

$$\int_a^b y\,dx \doteqdot \left(\frac{y_0}{2}+y_1+y_2+\ldots+y_{n-1}+\frac{y_n}{2}\right)h.$$

This is the *trapezoidal rule* for the approximate calculation of integrals.

14.5. Calculate, to two decimal places, approximations to (*a*) $\int_1^2 dx/x$ (take 5 intervals), (*b*) $\int_1^4 dx/x$ (take 6 intervals), and show that $\int_1^4 dx/x$ is approximately twice $\int_1^2 dx/x$.

14.6. [In order that the boat, when it is launched, may not float as shown in the figure, the designer has to calculate the position of the centre of buoyancy. This is the point through which the supporting pressure of the water may be regarded as acting,

and it is shown in hydrostatics that this point is the c.g. of the water displaced by the hull. Therefore to find the centre of buoyancy we must find the c.g. of a uniform solid coinciding with the under-water body of the yacht.]

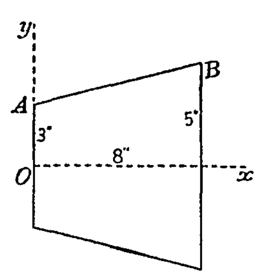
Show that if the area of the cross-section at distance $x$ ft. from the bow is $y$ sq. ft., the distance, $h$ ft., of the centre of buoyancy from the bow is given by

$$h\int_0^l y\,dx = \int_0^l xy\,dx,$$

where $l$ is the length of the yacht.

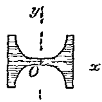
One of these integrals has already been evaluated approximately in 14.3 for the yacht *Paida*. Find the approximate value of the other by the trapezoidal rule and determine the distance of the centre of buoyancy from the bow.

15. (*a*) A bucket is 8 in. high and the radii of the base and top are 3 and 5 in. If the bucket is placed as in the figure, find the equation of $AB$ with respect to the axes marked. Find the volume of the bucket.




(*b*) If the height of the bucket is $h$, radii of base and top $a$, $b$, show that the volume is $\frac{1}{3}\pi h(a^2+ab+b^2)$.

(*c*) Use the result of (*b*) to find approximately the volume of a barrel if it is 6 ft. high and if the radii at the base, at 2 ft. 6 in. above the base and at the top, are 2, 3, 2 ft.

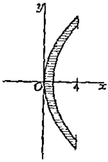


Ex. 5 Z 16

16 The diagram shows the cross section of a girder The curved boundary on the positive side of the $x$ axis is $y = \frac{1}{16}+7x^4$ from $x = -\frac{1}{2}$ to $x = +\frac{1}{2}$ and the girder is symmetrical about the $x$ axis The units of $x$ and $y$ are feet and the plane boundaries of the girder are given by $x = \frac{5}{8}$ and $x = -\frac{5}{8}$ If the density of the metal is 160 lb per cu ft find the weight of the girder per ft run

17 (a) Show that the curve $y = x^3$ divides the square formed by drawing parallels to the axes through (0 0) and (1 1) into two parts whose areas are in the ratio 1 3

(b) Show that if the curve is $y = x^n$ the ratio of the areas is 1 $n$

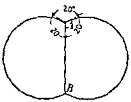


Ex 5 Z 18

18 Find the volume obtained by rotating the area between the curves $y^2 = 16x$ and $y^2 = 16(x-1)$ from $x = 0$ to $x = 4$ about the $x$ axis

19 Show that the tangent to $y = 8x^3+1$ at $x = -\frac{1}{2}$ meets the curve again at (1 9) Find the area enclosed between the tangent and the curve

20 When soap bubbles are blown it sometimes happens that a double bubble is formed consisting of two equal spherical portions separated by a plane circular film When this happens it is found that the three films make angles of 120° where they meet on the circle with $AB$ as diameter If the radii of the spheres are $a$ find the distance of the centre of each from $AB$ and show that the volume of the double bubble is $(9\pi a^3)/4$



Ex 5 Z 20

21 Find the volume generated by the rotation of $ay = x^2$ from (0 0) to ($a$ $a$) about (a) the $x$ axis (b) the $y$ axis (c) the line $y = a$

22 Show that the curve $3y = x^3-3x+4$ has a maximum at (−1 2) and passes through the point (2 2) Sketch the curve and find the area bounded by the curve and the straight line $y = 2$

23 Find the volume generated by the rotation of the part of the curve $y = x^2-x^3$ between $x = 0$ and $x = 1$ about (a) the $x$ axis (b) the line $y = 1$ Find the c g of the volume (a)

24 Any elementary text book on mechanics contains the following formulae for motion with constant acceleration $a$

(1) $v = u+at$ (2) $s = ut+\frac{1}{2}at^2$

$s$ and $v$ denote the distance from the starting point and the velocity

at time $t$, and $u$ is the velocity at the start. Prove the formulae as simply as you can, starting from the fact that the acceleration is $a$.

Find the corresponding formulae for motion with acceleration $kt^2$, where $k$ is a constant.

25. On a summer day the temperature $T°$ F. is given at $t$ hr. by the formula

$$T = \frac{t^3}{16} - \frac{3t^4}{256} + 70.$$

If $t = 0$ is noon, find the mean temperature from 8 a.m. to 8 p.m. Find also the temperature at 8 a.m. and 8 p.m., and the maximum temperature with the time at which it occurs.

What reason is there for supposing that clouds obscured the sun temporarily at noon?

26. Find the coordinates of the c.g. of the area bounded by the axes and the part of the curve $ay = a^2 - x^2$ for which $x$ and $y$ are positive. Illustrate your result with sketches drawn with the same axes for $a = 1, 2, 3$. Verify that the c.g.s of these areas are in line with the origin. Show that whatever the value of $a$, the c.g. is on the line $15y = 16x$.

27. The gradient function of a curve is $4(a-x)$, it meets the $x$-axis where $x = 3$, and the area in the first quadrant bounded by the curve and the axes is 18. Find $a$ and the equation of the curve. Find also the maximum width of the area parallel to the $y$-axis.

28. A curve of the form $y = a+bx+cx^2$ (where $a$, $b$, $c$ are constants to be found) is to cut the $x$-axis at $x = 2$ and have a maximum value of 1 at $x = 1$. Find $a$, $b$, $c$. Find where the curve cuts the $x$-axis again and find the volume obtained by rotating the part of the curve for which $y$ is positive about the $x$-axis.

29. Owing to the viscosity of water and the friction of the sides, the velocity of the water in a pipe is greatest at the centre. When the water is not flowing too fast, its velocity, $v$ f.p.s. at $x$ ft. from the centre of the pipe, is given by $v = k(r^2 - x^2)$, where $k$ is a constant, and $r$ ft. is the radius of the pipe.

(1) Find the value of $k$ for a pipe of 6-in. radius if the velocity of the water at the centre is 2 f.p.s.

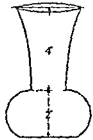
(2) Find the quantity of water, $Q$ cu. ft., flowing through this pipe per sec.

(3) Find a general formula for $Q$ in terms of $k$ and $r$.

(4) If $v_c$ is the velocity at the centre and $v_m$ is the mean velocity (the constant velocity over the whole section which would produce the same flow), show that $v_m = \frac{1}{2}v_c$.

30

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Distance from base (in ) | 0 | 0 5 | 1 | 1 5 | 2 | 3 | 4 | 5 | 6 |
| Area of cross section (sq in ) | 7 1 | 13 8 | 17 7 | 12 1 | 4 | 4 | 4 5 | 6 | 9 6 |




The pewter vase shown in the sketch has circular cross section and the area of its section at various distances from the base is given in the table above

Find the volume of the vase approximately

31 Find the mean value of the area of all the circles which can be drawn with radius not greater than a given length, $a$

# ANSWERS

*Note*: The answer is not given to a question in which a check is asked for.

## CHAPTER I

### EXERCISE 1.A (p. 11)

**1.** $\frac{1}{2}$. **2.** $\frac{1}{2}$. **3.** $-\frac{1}{2}$. **4.** 2. **5.** $-\frac{5}{3}$. **6.** $-2$. **7.** 2. **8.** (i) $\frac{2}{3}$; (ii) $-\frac{5}{9}$; (iii) $\frac{1}{2}$; (iv) $\frac{3}{11}$; (v) $-\frac{11}{13}$; (vi) $-5$. **9.** 1. **10.** $1\frac{3}{4}$. **11.** $1\frac{1}{2}$. **12.** 1. **13.** $-\frac{1}{3}$. **14.** $-\frac{3}{4}$. **15.** $\frac{3}{4}$. **16.** 0. **17.** 600 ft. **18.** Yes.

### EXERCISE 1.B (p. 14)

**1.** 78° 41′. **2.** 168° 41′. **3.** 108° 26′. **4.** 26° 34′. **5.** 135°. **6.** 0°. **7.** 90°. **8.** 49° 46′. **9.** 82° 53′. **10.** 11′. **11.** 173 ft. **12.** $\pm 1{\cdot}2$. **13.** 0·95. **14.** (*a*) $-0{\cdot}8$; (*b*) 141° 20′. **15.** 1 and 45°. **16.** (*a*) 51° 20′; (*b*) 38° 40′; (*c*) 77° 20′.

### EXERCISE 1.C (p. 16)

**1.** $2y = x-4$, $y = 3x-2$, $y = -4x-2$, $y = -x-2$. (*a*) $y = -4x-2$; (*b*) $2y = x-4$. **2.** The gradient of all the lines is 3. They are all parallel. $y-3x+k = 0$. **4.** $\frac{4}{3}$. **5.** $-3$. **6.** 2. **7.** $-\frac{1}{2}$. **8.** $\frac{1}{3}$. **9.** $-\frac{1}{2}$. **10.** 0. **11.** $-1$. **12.** $1\frac{1}{2}$. **13.** $-1\frac{1}{2}$. **14.** $3\frac{1}{3}$. **15.** 0°, 90°. **16.** (*a*) Lines parallel to the bisector of one of the angles between the axes; (*b*) lines parallel to the $x$-axis; (*c*) lines parallel to the $y$-axis. **17.** (*a*) $y = -x-2$; (*b*) $y = -2$. **18.** (*a*) $y = 4x+k$; (*b*) $y = -2x+k$; (*c*) $y = k$; (*d*) $y = \frac{1}{3}x+k$.

### EXERCISE 1.D (p. 17)

**1.** $-\frac{3}{4}$, $\frac{3}{4}$. Equally inclined. **2.** (*a*) Parallel; (*b*) equally inclined to axes; (*c*) equally inclined to axes. **3.** (*a*) $-3$; (*b*) $+2$. **4.** $\frac{1}{2}$. **5.** $\frac{5}{3}$, $-\frac{5}{3}$, $\frac{5}{3}$. **6.** (i) and (iii). **7.** (i) and (ii).

### EXERCISE 1.E (p. 20)

**1.** (1, 0), (0, 2), (0, 0); 3, 2, 1, 2. **2.** They are positive.
**3.** Third and fourth. **4.** First and third.
**5.** (*a*) Second; (*b*) first; (*c*) third; (*d*) fourth.
**6.** The speed is lowest at the top of the loop.
**7.** (*a*) (12, 2); (*b*) (6, 4); (*c*) they are the same; (*d*) they are the same; (*e*) (12, 4); (*f*) (6, $-4$); (*g*) $x$-coordinate$+4$, $y$-coordinate 0; (*h*) (4, 6); (*i*) (16, 8).
**8.**

| | *x-step* | *y-step* | | *x-step* | *y-step* |
|---|---|---|---|---|---|
| (*a*) | $+5$ | $+3$ | (*b*) | $+6$ | $-4$ |
| (*c*) | $+9$ | $-3$ | (*d*) | $+9$ | $+8$ |
| (*e*) | $-9$ | $+1$ | (*f*) | $-3$ | $-9$ |
| (*g*) | $+12$ | 0 | (*h*) | 0 | $-11$ |

9. $E$ step $18/\sqrt{2} = 12\ 7$ miles, $N$ step $2/\sqrt{2} = 1\ 4$ miles  10. (a) 0 7, rising, (b) 1 3, rising, (c) $-\frac{1}{2}$, falling, (d) $-2$, falling, (e) 0, parallel to $x$ axis, (f) 0 1, rising  11. (a) $(3, -4)$, (b) $(-3, 4)$  12 (a) $-1$, (b) 2, (c) $-4$, (d) 2  $c$ is the steepest line  13. (i) $2\frac{1}{2}$, (ii) $2\frac{1}{2}$  They lie on a straight line  15. 2  17. (a) 2 2, 65° 33′, (b) $-0\ 3$, 163° 18′  18. 90°, 53° 08′, 36° 52′.  19. (a) 5, 3, (b) $-4\frac{1}{3}$, $-2\frac{1}{2}$, (c) 1, $-\frac{1}{2}$, (d) 2, 0, (e) $\frac{1}{3}$, (f) 15  20. (2, 0) and (0, 3)
21. (4, 0) and (0, 3)  22. $x = 2, y = 3$  (2, 3)  23. (1, 1)
24 $(-2, 2)$, $(2\frac{1}{2}, 2)$, $(1, -1)$  25. (2, 2), $(-3, -1)$, $(3, -2)$
26 $(-1, 2)$ and (4, 5)

EXFRCISE 1 F (p 24)

1. (a) $y = 2x+k$, (b) $y = \frac{1}{2}x+k$, (c) $y = -\frac{3}{2}x+k$, (d) $y = -11x/7+k$
2 (i) $y-4x+3 = 0$, (ii) $2y+2x = 5$, (iii) $5y-3x = 16$, (iv) $2y+3x = 1$, (v) $y-x = 1$, (vi) $3y-10x = 1$, (vii) $y = 2$
3. $-\frac{3}{2}$  $2y-3x+5 = 0$  4. $3x+4y+7 = 0$
5. (i, a) $y$ axis (i, b) $x = 0$, (ii, a) $x$ axis (ii, b) $y = 0$, (iii, a) a bisector of the angle between the axes, (iii, b) $y = x$  6. Platform 4, $y = x+2$, $y = 2$, $x = 3$  7. 90°, $x = 1$  8. $x = 7$, $y+8 = 0$
9. $y+x = 3$, $y-x = 1$  10 $10y-8x = 9$, yes
11. (a) $2x+y = 4$, (b) $2x+y+4 = 0$
12. $2y+x = 5$, $2y-x+5 = 0$  13. $y-x+2 = 0$, $y+x+6 = 0$
14. $AD \equiv y-2x = 3$, $CD \equiv 3y-x = 4$, $D \equiv (-1, 1)$
15 Railway $3y+x = 7$  Road bridge $\equiv (1, 2)$  Distance from tunnel $= \sqrt{10}$ miles  Pond, nearer side, haystack, farther side  Bearings (a) N 53° E, (b) N 22° W
16 Red sector is bounded by $2y+x = 0$ and $y+3x = 0$  Its angle is 45°  Track of the trawler is $2y-x = 5$  Steamer and trawler enter red sector at $(-1\frac{3}{5}, 4\frac{4}{5})$ and $(-\frac{5}{7}, 2\frac{1}{7})$ respectively  The gun should be fired for the trawler

EXERCISE 1 G (p 27)

1. (2, 0), (0, 4)  2. (4, 0), $(0, -6)$  3 $(-1, 0)$, $(0, -1)$
4. $(-2, 0)$, (0, 1)  5. (2, 0), (0, 3)

EXERCISE 1 H (p 28)

2. $\frac{1}{5}x+\frac{1}{4}y = 1$  3. $-\frac{1}{3}x+\frac{1}{6}y = 1$ or $y-2x = 6$
4. $2x+y+4 = 0$  5. $2x+3y = 1$  6. $3y-4x = 2$  8. 20 ft
9. $5\frac{1}{4}$ ft  10. (a) $\frac{1}{5}x+\frac{1}{3}y = 1$, (b) $\frac{1}{5}x+\frac{1}{2}y+1 = 0$
11. $\frac{1}{3}x+\frac{1}{4}y = k$  12. $b = 30$ ft, 50 ft  13 $-b/a$

EXERCISE 1 J (p 30)

1. 63° 26′, 153° 26′.  2. (i) and (iii)
3. (a) $y = 2x$, (b) $3y+x = 0$, (c) $5y = 3x$
5. (a) $y-x-2 = 0$, (b) $3x+2y+2 = 0$, (c) $5y-2x+22 = 0$, (d) $x = 3$, (e) $y+3 = 0$  6. $2y = 3x$  9. The circle passes through $C$ and $D$
10. $y-x = d$  11. $y = 1$, $x = 3$
13. $y = 3x$, $3y+x = 0$, $y-3x = 10$, $3y+x = 10$  $(-2, 4)$

14. $3y-2x-10 = 0$, $2y+3x = 11$.
15. $y-2x = 5$, $2y+x = 0$. $(-0{\cdot}8, -1{\cdot}6)$, $(-2{\cdot}8, -0{\cdot}6)$, $(-2, 1)$, and $(0, 0)$.
16. (i) $\frac{1}{2}x+y = 1$; (iii) $C \equiv (1, 3)$, $D \equiv (3, 2)$; (iv) $3y = x+3$, $y+3x = 6$.
17. $4y-3x = 3$, $4y+x = 3$.

## EXERCISE 1.K (p. 32)

1. 8, 6, 10, $\frac{3}{4}$.
2. 6, $-6$, $6\sqrt{2} \doteqdot 8{\cdot}5$, $-1$.
3. $AB$, $AC$.

## EXERCISE 1.L (p. 34)

1. $(0, 3)$.
2. $(1, 0)$.
3. $(4, 4)$.
4. $(-2\frac{1}{2}, -3)$.
5. $(-1\frac{1}{2}, 1)$.
7. $x+y = 8$.
8. $5y-4x+9 = 0$.

## EXERCISE 1.M (p. 34)

2. 22 ft.
3. $3y+2x-2 = 0$, $3y+2x+2 = 0$. $A \equiv (4, -2)$, $B \equiv (8, -6)$, $C \equiv (6, -4)$.
4. $\frac{4}{3}$ and $-\frac{5}{8}$.
5. $y+3x = 10$, $3y-x = 20$, $y+3x = 0$. $(3, 1)$ and $(-2, 6)$.
6. $(9, 7)$.
7. 17.
8. $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -2)$.
9. $M \equiv (\frac{1}{2}, 3)$, $N \equiv (\frac{1}{2}, -2\frac{1}{2})$, $P \equiv (4, 1)$, $Q \equiv (-3, -\frac{1}{2})$, $X \equiv (\frac{3}{2}, 0)$, $Y \equiv (-\frac{1}{2}, \frac{1}{2})$.
10. $X \equiv (3, \frac{3}{2})$, $Y \equiv (2\frac{2}{3}, 2)$. $XY$ is $2y+3x = 12$.
11. 5.
12. $\sqrt{5}$, $(3, 3)$ and $(3, -1)$.

## EXERCISE 1.N (p. 36)

1. $x = 1$.
2. $x-2y+6 = 0$.
3. $y = x$.
4. $x+2y+1 = 0$.
5. $y = 2$.
6. $x+3 = 0$.
7. $y = 2x$.
8. $3x+5y = 18$.
9. $y-x = 0$, $y+x = 0$.
10. $y^2 = 4x$.
11. $x^2+y^2 = 1$.
12. $x^2+y^2 = 2$.
13. $x^2+y^2 = 25$.
14. $(x-1)^2+(y-2)^2 = 4$ which simplifies to $x^2+y^2-2x-4y+1 = 0$.
15. (*a*) circle; (*b*) circle.

## EXERCISE 1.P (p. 37)

1. $(0, 1)$, 2.
2. $(1, 1)$, 3.
3. $(-1, 3)$, 1.
4. $(0, 0)$, 6.
5. $(-1, -5)$, 2.
6. Circle.
7. Not a circle.
8. Not a circle.
9. Not a circle.
10. Circle.
11. $x^2+(y-2)^2 = 4$, simplifying to $x^2+y^2-4y = 0$.
12. $(x-1)^2+(y+1)^2 = 9$ simplifying to $x^2+y^2-2x+2y-7 = 0$.
13. $(x+1)^2+(y+2)^2 = \frac{9}{4}$ simplifying to $4x^2+4y^2+8x+16y+11 = 0$.
14. 1, 2) and $(1, -2)$.
15. $(4, 3)$ and $(-3, 4)$.
16. $x^2+y^2+8x = 0$. The locus may be drawn by noting that the centre of the circle is $(-4, 0)$ and its radius is 4.

## EXERCISE 1.Q (p. 38)

1. $y^2-4x+4 = 0$.
2. $x^2-4y+4 = 0$.
3. $y^2 = 4x$.
4. $y = x^2$.
5. $3x^2-y^2+8x-16 = 0$.
6. $3x^2+4y^2-64x+256 = 0$.
7. $xy = a^2$.
8. $\pi x^2+8xy = 288$.

## EXERCISE 1.R (p. 41)

1. $c = 0$.
2. (i) Parallel to the $x$-axis; (ii) parallel to the $y$-axis; (iii) passes through the origin; (iv) is the $y$-axis.
3. (*a*) $y = k$; (*b*) $x = k$; (*c*) $y = mx$ and $x = 0$ *or* $ax+by = 0$.

**4.** (i) $y = 5x$, (ii) $3x+2y = 0$ **5.** (a) 28 6, (b) 57 3, (c) 573
**6.** $x/a+y/b = 1$ **7.** $k = 2$ **8.** $l = 0$ **9.** $m = 2$, $k = 4$
**10.** $m = g$, $l = b$ **11.** $y = 0\ 62x+1\ 8$ **12** $P = 0\ 8W$
**13.** $P = 0\ 6W+10$ No **14.** $W = 3\ 4h-80$ 3 4, $-80$

EXERCISE 1 S (p 44)

**1.** $x^2+y^2-x-y = 0$, $(\frac{1}{2}, \frac{1}{2})$, $1/\sqrt{2}$ **2.** $x^2+y^2+6x-8y = 0$
**3.** 0 **4.** (a) Its centre lies on the $y$ axis, (b) it passes through the origin **5.** $x^2+y^2-10x = 0$, $(5, 0)$, 5
**6.** $x^2+y^2-4y-4x+3 = 0$, $(1, 0)$ **7.** $x^2+y^2-2x-2y-6 = 0$
**9.** $x^2+y^2+8y = 9$, $x^2+y^2-8y = 9$
**10** $x^2+y^2-8x-7y+12 = 0$, $(0, 4)$ **11.** $x^2+y^2 = 4$, $(1\ 2, 1\ 6)$
**12.** $x^2+y^2+2x+4y = 0$, $x^2+y^2-4x-2y = 0$ **13.** $(2, 1)$, $\frac{3}{2}$
**14.** $a = 1$ **15.** $x^2+y^2+4x = 0$
**16.** (a) No, (b) yes $a = 0$ gives $x^2+y^2+2x+4y = 0$
**17.** Centre the origin, radius 0, i e the origin

EXERCISE 1 T (p 46)

**1.** $x+2y = 5$ **2.** $x-2y = 10$ **3** $3y = 4x+2$
**4.** $3y-x-5 = 0$ **5.** $x+y = 0$ **6** $x-y-4 = 0$
**7.** $y+1 = 0$, $x = 2$ **10** $4x+3y = 50$, $4x-3y = 50$, $P \equiv (12\frac{1}{2}, 0)$, $PA = 7\frac{1}{2}$ **11.** $4x = 3y$, $3x+4y = 0$, $(0, 0)$ **12.** $a^2/x_1$, $a^2/y_1$

EXERCISE 1 U (p 48)

**1.** 1 1 **2.** 1 01

EXERCISE 1 V (p 49)

**1.** 5 **2.** 9 **3.** 2. **4.** 13 **5.** 67,500 sq yd
**6.** 5 7 acres nearly ($5\frac{15}{22}$ acres exactly) **7.** 232 sq ft

MISCELLANEOUS EXERCISE 1 X (p 49)

**1.** $y-2x+8 = 0$ **2.** $x = 5$ **3.** 122° **4.** $y+x = l$, 135°
**5** (a) and (c) **6.** (c) **7.** $(5, -2)$ **8.** $P \equiv (5, -1)$, $Q \equiv (4, -2\frac{1}{2})$ **9.** $2y = x$ **11.** $y+3x = 9$
**12.** $y-2x+5 = 0$ **13.** $3y+5x = 7$ **14** (b) is perpendicular to (a), (c) is parallel to (a) **15.** $y = x$ and $y+x = 2$
**16** $6x+5y = 60$ and $5y = 6x$ $(5, 6)$ **17.** $-3\ 6$ and 4
**18.** $12\frac{1}{2}$° **19.** $x+5y = 3$ and $5x-y+11 = 0$ **20.** $17\frac{1}{2}$ ft
**21.** $2y-x+5 = 0$ **22.** $y = 1$ **23.** $4y = 3x$, $(4, 3)$
**24.** (b) is parallel to (a), (c) is perpendicular to (a) **25.** $5y = 3x$, $5y+3x = 30$ **26.** $4y+5x = 4$ **27.** (a) 2, (b) $1\frac{1}{2}$
**28.** $(1, 2)$, $(-4, -1)$ **29.** $(0, -2)$, $y+2 = 0$ **30** The point on $x+2y = 0$ **31.** $4y+3x+22 = 0$ **32.** $y+2x = 4$
**33.** $2y-3x+4 = 0$ **34.** $y = 8$ **35.** $(-1, 3)$, 4
**36.** $y = 2-2x$ **37.** $a = 0$ **38.** $x = y+1$ **40.** $2y+x = 9$
**41.** $3y+2x+2 = 0$, $2y-3x = 16$ **42** $x-2y+9 = 0$
**43.** $x^2+y^2 = 50$ **44.** $xy = 100$ **45** (a) $x = 7$, $y = 5$, (b) $x+5y = 32$, $y = 5x-30$ **46** (a) 37°, (b) 90°, (c) 0°

47. (0, 0), (1, 2), (−1, 4), (−2, 2). **48.** $a = 2$, (−1, −1), 3.
49. $6\frac{1}{2}$. **50.** $(2, \frac{1}{2})$.
51. The equation of the circle is $x^2+y^2 = 4x+6y$.
52. (i) $k = 0$; (ii) $k = -14$. **53.** $x^2+y^2 = 36$; $y-2 = 0$, $y+10 = 0$. **55.** $a = 0$; (−1, 0), 1. **56.** $2x+3y = 21$.
57. $-3, \frac{1}{2}, -3, \frac{1}{2}$. **58.** $x^2+y^2+2y = 3$; (0, −1), 2.
60. $y+2x+5 = 0$. **61.** $y+4 = 0$. **62.** $x^2+y^2 = x+2y$; $(\frac{1}{2}, 1)$, $\sqrt{5}/2$. **63.** $2x = 3$. **64.** $x^2+y^2+2x-6y-6 = 0$.
65. $4y-5x = 3$; $8y-3x = 27$. **66.** $y+2 = 0$, $x = 4$. **67.** 7.
68. (5, 8), (−11, −4). **69.** $x^2+y^2-2x+4y-20 = 0$. 5.
70. $2y-3x = 1$.

## MISCELLANEOUS EXERCISE 1.Y (p. 53.)

1. $y = 3x$. $(-1\frac{1}{3}, -4)$. **2.** $5x+3y = 0$, $5y-3x = 21$.
3. In $x$-axis $x-2y = 1$, in $y$-axis $2y-x = 1$. **4.** $DE \equiv 2y-x = 2$, $EF \equiv x+y = 1$, $FD \equiv y-2x+2 = 0$. **5.** $26\frac{1}{2}°$, $63\frac{1}{2}°$, 90°.
6. (−9, 4), (3, −4), (6, −6) are on $3y+2x+6 = 0$. **7.** $y = 2x$, $3y = x$, $2y+x = 5$. $A \equiv (1, 2)$, $B \equiv (3, 1)$. **8.** (−5, 0), (−5, 10).
9. 1 ft. **10.** 13 yd. 1 ft. **11.** $(\frac{2}{3}, -1)$, $\frac{2}{3}$. $3y+4x = 3$.
12. $2y-4x = 5$, $4y+2x+5 = 0$; $(-1\frac{1}{2}, -\frac{1}{2})$. $OA = OB = OC = \frac{1}{2}\sqrt{10}$.
13. $3x-2y-4 = 0$. $Q$ and $R$. **14.** $8x = 17$. **15.** 2.
16. $(\frac{5}{2}, 3)$; $2\frac{1}{2}$; $x^2+y^2-5x-6y+9 = 0$. **17.** 5.
18. (−2, 4), (−5, 11). **19.** Parallel. $y-x+2 = 0$, $x = 1$, $y+x = 4$. $A \equiv (1, 3)$, $B \equiv (3, 1)$, $C \equiv (1, -1)$.
21. $20\frac{1}{2}$. **22.** $6\frac{12}{13}$ ft. ≑ 6 ft. 11 in. **23.** 26 ft., 28 ft.
24. Yes. **25.** 27 m., 105 m., 858 sq. m.
26. $A$; $x^2+y^2-2x-5y+1 = 0$. **27.** $x+2y = 10$; (2, 4) and (6, 2).
28. (1, 3), (−2, −2), (3, −5). **29.** (*a*) $7\frac{1}{2}$ ft.; (*b*) 7 ft.
30. $y+1 = 0$, $2y+12x = 7$; $K \equiv (\frac{3}{4}, -1)$;

$$KA^2 = KB^2 = KC^2 = KD^2 = \tfrac{185}{16}. \text{ Yes.}$$

32. $m = 2{\cdot}1$; the ball should be aimed at $(3\frac{1}{7}, 6)$.
33. $y-3x = 4$, $y = 2$; $(-\frac{2}{3}, 2)$. **34.** $y = 5x$, $13y-5x = 20$; $(\frac{1}{3}, \frac{5}{3})$; $11y+5x = 20$. **35.** $y = 23+0{\cdot}87x$.
36. 42; $7x+12y = 84$. **38.** $E = 0{\cdot}226+0{\cdot}05r$.

## MISCELLANEOUS EXERCISE 1.Z (p. 57)

2. 2.1. Beacon Hill. 2.2. Aldboro Point. 2.3. Marsh Farm. 2.4. 051142. 2.5. 082157. 2.6. 078161. 2.7. NW. end 051174; SE. end 058169. 2.9. 076162, approximately north. 2.10. 090020. 2.11. About N. 10° W.
4. Sides $\frac{2}{3}, -\frac{3}{2}, \frac{2}{3}, -\frac{3}{2}$; diagonals $5, -\frac{1}{5}$. **6.** $(-1, 1\frac{1}{2})$, $(-\frac{1}{2}, -2)$, $(3, -1\frac{1}{2})$. $14y-2x = 23$, $2y+14x+11 = 0$, $14y-2x+27 = 0$, $2y+14x = 39$. **7.** (*a*) (i) 6 in.; (ii) 12 in. **8.** $y+2x = 5$, $3y+x = 5$, $y-3x = 5$, $7y-x+5 = 0$. **9.** (3, 7).
10. $x^2+y^2-7x-2y-8 = 0$.
11. $x^2+y^2-2ax-2ay+a^2 = 0$; $x^2+y^2-2ax+2ay+a^2 = 0$. **12.** (2, 1).
14. (1, 4), (−3, 1), (2, 0); $4y-x = 7$, $y-5x+1 = 0$. $(\frac{11}{19}, \frac{36}{19})$.
15. $bx-cy = ab$, $ax-cy = ab$.

**16.** $x^2(1-k^2)+y^2(1-k^2)-2x(1+k^2)+1-k^2=0$ When $k=1$, the equation reduces to the equation of the perpendicular bisector of $(-1, 0)$, $(1, 0)$
**17.** (i) $(1, 3)$, (ii) $(6-x_1, 8-y_1)$, (iii) $y+2x=15$
**18.** $15x+4y=140$, $25\frac{5}{8}$ ft above the ground
**19.** (i) $x/2x_1+y/2y_1=1$, (ii) $2xy=a^2$, (iii) $4x^2+4y^2=b^2$
**20.** $AD \equiv 3y-2x-1=0$, $CF \equiv x-1=0$, $G \equiv (1, 1)$, $X \equiv (-2, 1)$, $3y-2x=7$, $x+2=0$ **21.** $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$, $y=x$. **22.** 17, $y-3x=2$ **23.** (a) 192, (b) 1 **24.** (6.7, 8) **28.** (5, −1), (−2, −2) **29.** If £$P$ is the price of a yacht of $T$ tons, the law is approx $P=66(T-1)$, no, the price is too high **30.** If £$P$ = price, $H$ = horse power, the laws are $P=4.5H+38$ and $P=3.9H+43$ approx 8 H P **31.** If $S$ cwt = breaking strain and $W$ lb = weight of 100 fathoms, $S=2+0.32W$ approx
**32** $CM_1 \equiv 2y+3x-5=0$, $M_1E \equiv 2y-3x+5=0$,
$CN_2 \equiv 4y-3x-1=0$, $N_2M_2 \equiv 4y+3x-1=0$,
$M_2E \equiv 4y-3x+1=0$, $y=x$
**33.21.** Order 5, $\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$, order 7, $\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$ **33.23** $\frac{1}{2}, \frac{5}{9}, \frac{9}{16}, \frac{4}{7}, \frac{3}{5}$ **33.31.** At a point in cushion 3, 2 ft from D **33.32.** At a point in cushion 2, 1.5 ft from $C$ **33.33.** At a point in cushion 2, 3 ft from $C$, in or on the boundary of the triangle $ACD$

## CHAPTER II

### EXERCISE 2 A (p 70)

(a) 5 (b) 4.1 (c) 3 (d) 3.9

### EXERCISE 2 B (p 72)

**2.** (a) 4.01, (b) 3.99 Between 3.99 and 4.01
**3** (a) 4.001, (b) 3.999 4.00

### EXERCISE 2 C (p 73)

**1.** (a) $8+h$, (b) $8-h$, (c) 8 **2.** $6+3h$, 6

### EXERCISE 2 D (p 74)

**1.** (a) 5, (b) 4.01, (c) 3.9, (d) 3.999 4 **2.** 2 **3** 3 **4.** 10
**5.** 2 **6.** 2 **7.** 8 **8.** 2, 8, 5 **13.** (a) 4, (b) 4

### EXERCISE 2 E (p 76)

**1.** $y=2x$ **2.** $y+2x+1=0$, $y-6x+9=0$
**3.** $y-2x+4=0$, $4y+2x+1=0$ **4.** $y-2x=0$, $y+2x=4$

### EXERCISE 2 F (p 78)

**1.** 8, −2, 2, $-\frac{1}{2}$ **2.** 4.5, −21, 1.5, −7 **3** $-5h$, −5
**4.** −3.5 **5.** 8, $x=-2$, $x>-2$ **6.** $x<6$ **7.** $6\frac{1}{2}$, $x>3$
**8.** 4, (a) 11, (b) 27. **9.** $-\frac{3}{2}$, (a) $15\frac{1}{2}$, (b) 22

10. 2·67 lb. per ft., 1·78 lb. 11. 0·005 in. per ° F., 0·425 in.
12. 75 gal. per hr., 800 gal. 13. 0·003 H.P. per gal.

EXERCISE 2.G (p. 80)

1. (i) $E$; (ii) 2,500, 3,000, −4,000, −2,000, $C$; (iii) $C$, D; (iv) 50 yd. $E$., 100 yd. W., 200 yd. E., 20 yd. E.; (v) $A$, $D$, $B$, $C$; (vi) 28 yd. W.
2. 60 m.p.h., 1 hr. 27 min. 3. 50 m.p.h.
4. 45 m.p.h. (22 yd. per sec.). 5. −8 f.p.s.
6. $8\frac{1}{2}$ f.p.s. 7. $4\frac{1}{2}$ f.p.s. 8. 502 ft. per min.

EXERCISE 2.H (p. 82)

1. (i) $100+10h$; (ii, $a$) 101; (ii, $b$) 100·1; (ii, $c$) 100·01; (iii) 100.
2. $12+3h$, 12. 3. $4+\frac{1}{2}h$, 4. 4. $-6-h$, $-6$. 5. 12·6 lb. per f.p.s.; 10·8 lb. per f.p.s. 6. (i) 4; (ii) 4; (v) tangent to the curve at (4, 8). 7. 1, $y = x-2{\cdot}5$. 8. −4, decreasing.
9. ($a$) Cod, about 16 cm. per yr.; mullet, about 7 cm. per yr. ($b$) Cod, about 10·5 cm. per yr.; mullet, 2 cm. per yr. ($c$) 4 yr. ($d$) Cod, 14·2 cm. per yr.; mullet, 6·8 cm. per yr. 10. ($a$) 1·2 in. per 1,000 ft., ($b$) 0·8 in. per 1,000 ft. Between 0·9 and 1·0 in. per 1,000 ft.

EXERCISE 2.J (p. 84)

1. 10, 20. 2. 1,000 yd. per min.
4. ($a$) 5·5 yd. per sec.; ($b$) 9·5 yd. per sec.; ($c$) no; ($d$) 11·4 sec.
5. (i) 4 f.p.s.; (ii) 2 f.p.s.; (iii) 2·5 sec.; (iv) −10 f.p.s.
6. ($a$) 8·5 yd. per sec.; ($b$) 1·5 yd. per sec.; ($c$) 3·5 yd. per sec.

EXERCISE 2.K (p. 86)

1. $2x/5$, $y-2x+5 = 0$. 3. $y-4x+4 = 0$, $y-8x+16 = 0$.
4. 5, (−0·1, 0·01), $100y+20x+1 = 0$. 5. $x = -1$.
6. $0{\cdot}12x$, 2·4 lb. wt. per m.p.h., 50 m.p.h.
7. $32x$; (i) 96 f.p.s.; (ii) $v = 32t$; (iii) $12\frac{1}{2}$ sec.; (iv) 4 sec.; (v) 128 f.p.s.
8. $8x$.

EXERCISE 2.L (p. 88)

1. $4x^3$. 2. $5x^4$. 3. 1. 4. 0.

5.

| Function | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x$ | 1 |
|---|---|---|---|---|---|---|
| Gradient function | $5x^4$ | $4x^3$ | $3x^2$ | $2x$ | 1 | 0 |

EXERCISE 2.M (p. 89)

1. $-1/x^2$. 2. $-3/x^4$. 3. $6x^5$, $7x^6$. 4. Yes, e.g. gradient function of $x^{-1} = -1 . x^{-2} = -1/x^2$. 5. $30x^2$. 6. $2x+3x^2$; by adding them.
7. $1+1/x^2$; by subtracting the gradient function of $1/x$ from the gradient function of $x$. 8. $2-6x$.

## EXERCISE 2 N (p 90)

1. $6x$ 2 $\frac{1}{2}x^2$ 3. $0$ 4. $56x^6$ 5. $x^4$. 6. $-4/x^3$
7. $-1/x^3$ 8. $4/x^5$ 9. $1/x^4$ 10 $2-2x$ 11. $8$
12. $12x^2$ 13. $1+\frac{1}{2}x$ 14. $-6x^2$ 15. $2-9x^2$
16. $x+x^2$ 17. $x-2x^3$ 18. $(4x-1)/5$ 19. $2x^3-4$
20. $2/x^3$ 21. $-9/x^4$ 22 $-1/x^5$ 23. $1+1/x^2$
24. $2x-2/x^3$ 25. $6x^2+6/x^3$ 26 $6x-6/x^4$ 27. $3+8x$
28. $2x+2$ 29. $1-1/x^2$ 30 $1+2x+3x^2$ 31. $-1/x^2-1$
32. $4-1/x^2$ 33. $3x^2-4$ 34. $6x^5-6x^2$ 35. $10x+6/x^4$
36. $9x^2+8x+5$ 37. $x^4-2$ 38. $9x^8+9x^{-10}$ 39. $8x-18/x^3$
40. $\pi(2x-4x^3) = 2\pi x(1-2x^2)$ 41. $3\pi(x^2+1/x^2)$
42. $v = 20t-4$ 43. $v = \frac{1}{2}t-4$ 44. $v = t-1$
45. $v = 12(t^2-1)$ 46. (*a*) $-11$ 25 tons per ft, (*b*) $-0$ 45 tons per ft
47. (*a*) $-3$, 24 Opposite directions, (*c*) $t = 2$ $(-4, 0)$, (*d*) $t = 3$ 9
48. (i) $v = 27-\frac{3}{4}t^2$, (ii) 24 ft per min, $-48$ ft per min, (iii) 27 ft per min, (iv) 6 min., (v) 27, 6, 108, $-54$ 49. (ii) $v = 80-32t$, (iii) 48 f p s, moving upwards, $-16$ f p s, moving downwards, (iv) 80 f p s, (v) $2\frac{1}{2}$ sec, (vi) 100 ft 50. $v = -36-32t$, $-164$ f p s, i e 164 f p s downwards 51. $(3, 9)$, $y-6x+9 = 0$
52. $3y-2x+2 = 0$ 53. (*a*) $y+2x+4 = 0$, (*b*) $y = 1$
54. $(-\frac{1}{2}, \frac{1}{6})$ 55. $3y+x+1 = 0$

## EXERCISE 2 P (p 95)

1.  2.  3.  4. 

5.  6.  7. 

8. (*b*) 9. (*a*) 10 (*c*) 11. (*a*), (*b*), (*c*) 12 (*a*)
13. (*c*) 14 (*b*) 15. (*c*) 16. (*b*)

## EXERCISE 2 Q (p 98)

3. (*a*) $y = x$, (*b*) $y = -1/x$

## EXERCISE 2.R (p 98)

For each curve, $y = 2x$ and the tangents at $x = 2$ are parallel with gradient 4 $c = -3$

## EXERCISE 2 S (p 99)

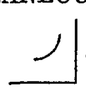
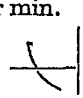
1. $y = x^3+c$ 2. $y = x^4+c$ 3 $y = x^5+c$ 4. $y = x+c$.
5 $y = 2x^3+c$ 6. $y = -x^2+c$ 7. $y = x+x^2+c$
8 $y = 3x-x^2+c$ 9 $y = \frac{1}{2}x^2+c$ 10. $y = x+\frac{1}{2}x^2+c$
11. $y = 2x-\frac{1}{2}x^2+c$ 12. $y = \frac{1}{4}x^2+c$ 13 $y = c-\frac{1}{8}x^2$
14. $y = \frac{1}{5}x^2+c$ 15. $y = x-2x^2/3+c$ 16 $y = 0$ $2x-0$ $3x^2+c$
17. $y = \frac{2}{3}x^3+c$ 18. $y = \frac{1}{2}x^3+c$ 19. $y = x+x^3+c$
20. $y = x+\frac{1}{3}x^3+c$ 21. $y = \frac{1}{6}x^3+c$ 22. $y = c-\frac{1}{12}x^3$
23. $y = \frac{1}{6}x^3+c$ 24. $y = \frac{3}{4}x^2+x^3+c$ 25. $y = \frac{1}{10}x^2-\frac{1}{15}x^3+c$

26. $y = x-\frac{1}{2}x^2+\frac{1}{3}x^3+c$. 27. $y = \frac{1}{4}x^4+c$. 28. $y = \frac{1}{2}x^4+x^3+c$.
29. $y = (x^4+2x^2)/20+c$. 30. $y = x^4+2x^3+c$.
31. $y = \frac{1}{6}x^3-2x^2+2x+c$. 32. $y = c-1/x$. 33. $y = c+2/x$.
34. $y = c-1/2x$. 35. $y = \frac{1}{2}x^2+1/x+c$. 36. $y = 1/x^2+c$.
37. $y = \frac{1}{3}x^3-1/x+c$. 38. $y = \frac{1}{4}x^4+1/2x^2+c$.
39. $y = \frac{3}{2}x^2+x-2/x+c$. 40. $y = 2x^2+4/x+c$.
41. $y = c-1/8x^2$. 42. $y = \frac{5}{3}x^3+10/x+c$. 43. $y = \frac{1}{2}x^2+1/x^2+c$.
44. $y = 1/x-1/2x^2+c$.

## EXERCISE 2.T (p. 100)

1. $y = x^2+1$. 2. $2y = x^2+6$. 3. $y = 2x^2+5x+3$.
4. $y = 2x^2$. 5. $y = 1+x-x^3$. 6. $y = x^4-1$. 7. $s = t-t^2$.
8. $s = 2t^2-2t/3$. 9. $s = 20t$. 10. $s = t^3-5t$.
11. $s = 3t^3-36t$. 12. $s = 40t+16t^2$. 13. $s = 2t^4+t$.
14. $s = t+\frac{1}{4}t^3$. 15. $s = 3t^2+t^3$. 16. $s = t-2t^2+\frac{4}{3}t^3$.
17. 11, 3. 18. 10. 19. 100, 20, 0. 20. $2y = 3x-6$.
21. $3y = x^3+3$. 22. $4y = x^2$. 23. $xy = 10x-1$.
24. $2y = 8+2x-x^2$. 25. 5 ft. 26. 8 yd. 27. $3\frac{7}{8}$ yd.
28. 13 ft. 29. 9 ft. 30. 48 cm. 31. 4.
32. $y = 2+3x-2x^2$; $x$-axis (2, 0), $(-\frac{1}{2}, 0)$; $y$-axis (0, 2); 36.
33. $y = x-\frac{1}{6}x^2+c$. 34. $y = 2x^3+c$. 35. $y = -6/x+c$.
36. $y = \frac{5}{4}x^4+c$. 37. $y = 2/x^2+c$. 38. $y = -1/16x^2+c$.
39. $y = \frac{1}{2}x+c$. 40. $y = 5x^2/4+c$. 41. $y = 3x/2-\frac{1}{3}x^2+c$.
42. $y = 9x-10x^3/9+c$. 43. $y = x^2+1/2x+c$.
44. $y = x^3/9+3/x+c$. 45. $y = 2x^3-2x^2+5x+c$.
46. $y = \frac{1}{3}x^3-2x-1/x+c$. 47. $y = \frac{1}{3}x^3-\frac{3}{2}x^2+2x+c$.
48. $y = \frac{1}{3}x^3+x^2+x+c$. 49. $y = c-4x+\frac{5}{2}x^2-\frac{1}{3}x^3$.
50. $y = x-1/x+c$. 51. $y = c-8/x-4x$.
52. $y = \frac{3}{8}x^4+x^3+\frac{3}{4}x^2+c$. 53. $y = 2x^3-3x^2+2$. 54. 98 yd.
55. 16·8 ft.

## MISCELLANEOUS EXERCISE 2.X (p. 102)

1. $3x^2-3$, 9, 1. 2. . 3. $y+8x+16 = 0$.
4. $3y = x^3-3x^2+3$. 5. $1-2x$. 6. $\frac{1}{8}$.
7. 3,303, 103 ft. per min. 9. $2x-8/x^3$. 10. $v = \pi(\frac{2}{3}h^3-h^4/4+\frac{1}{12})$.
11. $2x-2$. 12. . 13. The $y$-axis. 14. (1, 4), $(-1, -4)$.
15. $y = x^3-2x^2+x-4$. 16. $5x^4-5/x^6$. 17. $4x$.
18. $y+x = 0$, $y-2x+2 = 0$, $y-2x = 2$. 19. $y = \frac{1}{2}x^2+1/2x^2+c$.
20. $v = 20/t^2$, rising; 996 ft., 0·8 f.p.s. 21. $(1, 4\frac{5}{6})$, $(-2, -8\frac{2}{3})$.
22. $\pi-8x^3$. 23. $1+2x$. 24. $48\frac{3}{4}$ ft.
25. (a) 1, (b) 8, (c) 0. $x = -3\frac{1}{3}$ or 4. 26. 4 and 1. 28. $4+8x+3x^2$.
29. $(\frac{1}{3}, 2)$; $y = 2x^2+\frac{3}{2}$. 30. 14°.
31. 4 sec., 24 ft. 32. $k = 4$, $y = 2x^2+7$.
33. $y = \frac{1}{2}x^2+1/x+c$, $y = 2$. 34. (a) 125 ft., (b) 500 ft.

35. (a) $2x+3$, (b) $3x^2+6x+2$ 36. $30\pi = 94\ 2$ sq in per in
37. $(\frac{1}{3}, 3)$, $(-2, 3)$, $3y-21x = 2$, $y+7x+11 = 0$
38. $y = 1-1/3x^3$ 39 $\frac{1}{2}$ f p.s, (a) 1 sec, (b) 9 sec
40. $y-4x+2 = 0$, $9y-36x = 14$ 41. $3y = 6+3x-2x^3$.
42. $1+6x-6x^2$. 43. $29\frac{1}{3}$ 45. $y = 2x^3-5x-7$
46. $2y+2x = 1$, $8y-25x+125 = 0$ 47. $-9\ 2$
48. $-32$, $36$ 49. $-1$, $-4$ 50. $\frac{1}{2}$ or $\frac{4}{5}$

## MISCELLANEOUS EXERCISE 2 Y (p 105)

1. $18x^3$ 2. $1+x$ 3. $2x+2/x^2$ 4. $1-2/x^2$.
5. $\frac{1}{2}-2/x^2$ 6. $10x^9-10/x^{11}$ 7. $10\pi x(4-3x)$
8. $8\pi x(1+2x^2)$ 9. $2(x+k)$ 10. $3x^2+10x+4$ 11. $\frac{1}{4}x^4+c$.
12. $x^5/25+c$ 13. $c+x+x^2+x^3$ 14. $\frac{4}{3}\pi x^3+c$
15. $c+x+3x^2+3x^3$ 16 $x+1/x+c$ 17. $x^3+12/x+c$
18. $x^4+x-2/x^2+c$ 19. $\frac{1}{3}x^3+kx^2+k^2x+c$ 20. $c+3/x-15/2x^2$.
21. $v = 20t^3+2t$ 22. $v = \frac{1}{2}t-2$ 23. $s = t^5-20t^3+180t$
24. $s = \frac{2}{3}t^3+t^2-1/t-\frac{1}{3}$ 25. $3y+9x = x^3$
26. $y = 2$ or $y = -2$ 27. (a) $y-x = 3$, (b) $y+x = 3$
28. $y-6x+8 = 0$, $y-6x = 0$ 29. (a) $3+h$, (b) 3 01, (c) 3 001, 3 f p s
30. (i) At $t = 4$ sec, (ii) $t = \sqrt{48} = 4\sqrt{3} = 6\ 9$ sec, (iii) 88, 115 ft, (iv) 88, 141 ft, (v) 11 ft, (vi) 32 f p s, (vii) 21 f p s
31. (a) 128 f p s upwards, (b) 400 ft, (c) at $t = 9$ sec with velocity $-160$ f p s, (d) 144 ft 32. $v = 0$ when $t = 5$ sec and $s = -25$ ft 10 f p s 33. (a) $(t-t^3/9)$ ft, (b) 3 sec, $-2$ f p.s
34. $2\frac{2}{3}$ and 4 ft 35. 10 ft, 5 ft 36. (i) (a) Sunrise $-3$ min. per wk, sunset 9 min per wk, (b) sunrise $-7$ min per wk, sunset 11 min per wk (ii) 12 min per wk and 18 min per wk (iii) $-5$ min per wk and 10 min per wk 37. (a) $10\frac{1}{2}'$ per min, (b) 0 6′ per min, $-1'$ per min, (c) about 56° $59\frac{1}{2}'$ at 12 $07\frac{1}{2}$
38 About 21 m p h, 35 m p h 39. (a) 15 f p.s downwards, 11 f p s upwards, (b) 30 f p s, 20 f p s 40. $-\frac{1}{4}$.
41. $z = \frac{3}{4}x^4-2x^3+4$ 42. 0 368 mm per mm, 0 59 mm per mm, or about 1 6 times the rate for the stag beetle 43. $A \equiv (-\frac{1}{2}, 2)$, $B \equiv (1, 2)$, tangent at $A$, $2y+6x = 1$, tangent at $B$, $y-3x+1 = 0$, $4x = 1$ 44. $y+3x = 4$, $y+6x = 8$
45. $y-2x = 2$, $y-6x+10 = 0$ 46. $6y = 5x+3$, $2y-7x+15 = 0$
47. $(4, 0)$, $(1, 3)$, $(-1, 5)$, $y+16x = 64$, $y-5x+2 = 0$, $y+11x+6 = 0$, $(-\frac{1}{4}, -\frac{13}{4})$, $(\frac{22}{7}, \frac{96}{7})$, $(14, -160)$ 48. $0$, $-2a$, $18a$, $a = \frac{1}{3}$, $y = 0$, $3y+x = 1$, $y-3x+9 = 0$ 0 6 sq units
49. $2y+3x = 4$, $2y+3x+4 = 0$ 50. 5° 43 51. 26° 34′.

52.

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $y$ | $8$ | $4$ | $0$ | $-4$ | $-8$ |
| $g$ | $20$ | $-10$ | $0$ | $-10$ | $20$ |

The graph is not straight

# CHAPTER III

## EXERCISE 3.A (p. 111)

**1.** (*a*) 105; (*b*) 84. **2.** 0·0008. **3.** 0·0001.
**4.** (*a*) −0·0372; (*b*) −0·0080. **5.** 5·762. **6.** 0·1923.
**7.** 9·899. **9.** 2·5. **10.** 1·25. **11.** $\delta y = \delta x + 2x\,\delta x + (\delta x)^2$.
**12.** $\delta y = -6x\,\delta x - 3(\delta x)^2$. **13.** $\Delta r = -\dfrac{100\Delta s}{s(s+\Delta s)}$, −0·099.
**14.** $\delta r = -\dfrac{\delta n}{n(n+\delta n)}$, (*a*) $-\frac{1}{3}$, (*b*) $\frac{1}{90}$. **16.** $\delta y/\delta x$.

## EXERCISE 3.B (p. 113)

**1.** $8x$, $dy = 8x\,dx$. **2.** $dy = 3x^2\,dx$. **3.** $dy = -1/x^2\,dx$.
**4.** $dy = (2-16x)\,dx$. **5.** $dy = 2x\,dx + \frac{1}{2}x^2\,dx$. **6.** $dy = 2x^3\,dx$.
**7.** $\delta y = 8\,\delta x + 2(\delta x)^2$, $dy = 8\,dx$; 10, 8.
**8.** $\delta y = -\delta x/(1+\delta x)$, $dy = -dx$; $-\frac{1}{11}$, $-\frac{1}{10}$.
**9.** $\delta y = 20\,\delta x + 10(\delta x)^2$, $dy = 20\,dx$, $\delta y - dy = 10(\delta x)^2$. (i) 0·1; (ii) 0·001.

**12.**

| $\delta x$ | 0·2 | 0·1 | 0·01 |
|---|---|---|---|
| $\delta y$ | 1·64 | 0·81 | 0·0801 |
| $dy$ | 1·6 | 0·8 | 0·08 |
| $\delta y - dy$ | 0·04 | 0·01 | 0·0001 |

**13.**

| $\delta x$ | 0·5 | 0·3 | 0·1 | 0·01 |
|---|---|---|---|---|
| $dy$ | 5 | 3 | 1 | 0·1 |
| $\delta y - dy$ | 1·25 | 0·45 | 0·05 | 0·0005 |

**14.**

| $\delta x$ | 0·1 | 0·5 | 1 | 2 |
|---|---|---|---|---|
| $dy$ | −0·2 | −1·0 | −2 | −4 |
| $\delta y - dy$ | 0·01 | 0·25 | 1 | 4 |

**15.** $-(\delta x)^2$. **16.** $dy = 2(x-1)\,dx$, 25.
**18.** $dy = 3x^2\,dx + 4x\,dx + dx$. **19.** $dy = (1-4/x^2)\,dx$.
**20.** $dy = 3x^2\,dx - 8x\,dx + 10\,dx$. **21.** $dy = (2x + 4/x^3)\,dx$.

## EXERCISE 3.C (p. 117)

**1.** (i) 3·64; (ii) 3; (iii) 1·6, 0·128.

**2.**

| $\delta x$ | 1 | 0·1 | 0·01 |
|---|---|---|---|
| $\delta y$ | 7 | 0·331 | 0·030301 |
| $dy$ | 3 | 0·3 | 0·03 |
| $\delta y - dy$ | 4 | 0·031 | 0·000301 |

$1{\cdot}1^3 \doteqdot 1{\cdot}3$, error 0·031
$1{\cdot}01^3 \doteqdot 1{\cdot}03$, error 0·000301.

3. 16 lb, 17 lb 4. 9 lb wt 5. A decrease of $1\frac{1}{4}$ tons
6. 10,400 7. 69,000 8. 888 9. A decrease of 0.1 ft candles.
10. 4.3

## EXERCISE 3 D (p. 120)

2. (a) Within about 0.23, i.e. to the nearest unit, (b) within about 0.015, i.e. to the first decimal figure 3. (i) $v = 900/t$, (ii) $\pm 2$ m.p.h.
4. 257.1 m.p.h. [The error is easily seen to be numerically smaller than $\frac{1}{10}$ m.p.h.] 5. Between 29.4 and 30.6 f.p.s. 6. $\pm 2.56$.
8. 400,000 sq. miles, $2.0 \times 10^6$ sq. miles

## EXERCISE 3 E (p. 122)

1. $16x-5$, 43 2. 8, $-\frac{1}{3}$ 3. $12t^2$, 75 4. 94
5. $y = x$ 6. $8y-4x = 51$ 7. (a) $y+3x = 3$, $3y-x+1 = 0$, (b) $y = 1$, $x = 0$ (1, 0), $(\frac{2}{3}, 1)$, (0, 1), $(0, -\frac{1}{3})$, $(\sqrt{5})/3$
8. $y-4x+4 = 0$, $4y+x = 18$ 9. $\frac{25}{24}$

10.

| | | | |
|---|---|---|---|
| $\delta x$ | 0.5 | 0.1 | 0.01 |
| $\delta y$ | 1.25 | 0.21 | 0.0201 |
| $dy$ | 1 | 0.2 | 0.02 |
| $\delta y/\delta x$ | 2.5 | 2.1 | 2.01 |
| $dy/dx$ | 2 | 2 | 2 |

(a) 2, (b) 2.01
11. $v = 3t^2-6t+2$, $ds/dt$, $-1$
12. (a) 36 f.p.s., rising, (b) $-28$ f.p.s., falling
13. (a) $7\pi/4$ sq. in. per in., (b) $\frac{3}{4}\pi$ sq. in. per in.
14. $y-x = 2$, $y-x+2 = 0$ 15. $12y-8x = 35$, $12y-8x+35 = 0$
16. $y-2x = 0$, $6y+3x = 5$, $6y+3x+5 = 0$

## EXERCISE 3 F (p. 124)

1. $y$ increases, (a) $x > 0$, (b) $x < 0$ 4. $-1 < x < 1$
5. Negative values of $x$ 9. $-1$, 1, positive 10. $y = 1$, $-$

## EXERCISE 3 G (p. 128)

1. 4, yes 2. $-1$, minimum 3. 2, maximum
4. 10, minimum 5. 0, minimum. 6. 2, maximum.
7. minimum 8. $a$, $c$, $e$ (i) $a$, (ii) $e$
9. 2, minimum, $-2$, maximum 10. $1\frac{1}{2}$, minimum
11. 0, maximum, $-4$, minimum 12. 3, maximum, $-1$, minimum
13. 7, maximum, 3, minimum 14. 8, maximum, $-8$, minimum
15. 18, minimum 16. $\frac{8}{3}$, maximum, 24 minimum
17. 2 minimum values, each 2, at $x = 1$ and $-1$

## EXERCISE 3 H (p. 129)

1. $3x^2-3$, $6x$ 5. 2 6. $-2/x^3$ 7. $-12\pi x^2$
8. $g = 40x^3-16x$, $dg/dx = 120x^2-16$
9. $g = 12x+4/x^2$, $dg/dx = 12-8/x^3$
10. $g = 2$, $dg/dx = 0$ 11. $g = 4x^3-2$ $dg/dx = 12x^2$

12. $g = 3x^2-6x+2$, $dg/dx = 6x-6$.
13. $g = -1/4x^2+\frac{1}{4}$, $dg/dx = 1/2x^3$.
14. $g = -2/x^3-2/x^2$, $dg/dx = 6/x^4+4/x^3$.

## EXERCISE 3.J (p. 130)

1. $24(5x^2+1)$. 2. $2(x^4-1)$. 3. $9x^8+2x^3$.
4. $6/x^4+20/x^6$. 5. $8+1/x^2$. 6. $6x+2/x^3$. 7. $12x^2+8$.
8. $20ax^3$. 9. $\frac{1}{2}(x^4+1/x^2)$. 10. $2a$.
11. (*a*) $dy/dx$, $d^2y/dx^2$; (*b*) $dW/dl$, $d^2W/dl^2$. 12. 2.
13. 42 lb. per mile. 14. (*a*) $-2k/l^3$; (*b*) $6k/l^4$.
15. '9 a.m. Sounded the well. Level of water *rising*. Started the pumps. 10 a.m. Level of water *falling* at a *constant rate*. 11 a.m. Level of water *rising* at an *increasing rate*. . . .' 16. $dy/dx = (x^2+1)/2$.
17. $dy/dx = 2(x^2+x-1)$. 18. $y = \frac{1}{2}(x^4-2x+3)$.
19. $y = 1/2x+5x/2+3$.

## EXERCISE 3.K (p. 134)

1. Minimum, $y = -88$. 2. Maximum, $y = \frac{1}{2}$. 3. Neither.
4. Maximum, $y = 0$. 5. Minimum, $y = 2$.
6. 24, maximum; $-3$, minimum. 7. 2, maximum.
8. 45, maximum; $-9$, minimum. 9. 0, maximum; $-10\frac{2}{3}$, minimum.
10. 6, maximum; $-12\frac{14}{27}$, minimum. 11. 3, minimum.
12. $\frac{3}{4}$, maximum; 0, minimum.
13. 0·16, maximum; $-0{\cdot}16$, minimum.
14. Two minimum values, each 2, at $x = 2$ and $x = -2$.
15. $\frac{5}{9}$, maximum; $-3$, minimum.
16. 0, maximum; $-16$, minimum. 17. 27, minimum.
18. $-9^9\times10^{-10}$, maximum.

## EXERCISE 3.L (p. 136)

1. 1. 2. (*a*) 40, 40; (*b*) 40, 40; (*c*) $\frac{1}{2}$, $79\frac{1}{2}$. 3. $21\frac{1}{3}$ cu. in.
4. $1000\pi$ cu. in. 5. $4\sqrt{3} \doteqdot 6{\cdot}9$ ft. 7. $v = x(1-2x)^2$; 2 in.
8. $x = 1$, $y = 2$. 9. 8 in., 16 in., 24 in. 11. (i) 5,000 sq. yd.
12. $y = 4x/3$. 13. $(14/v+v/56)$ pence; 28 m.p.h.
15. Base 8 in. × 24 in., height 6 in.

## EXERCISE 3.M (p. 141)

1. Point of inflexion $(-1, 11)$. Turning-points $(-3, 27)$ (maximum); $(1, -5)$ (minimum).
2. (1) Yes; (2) no. 3. (2, 16).
4. Point of inflexion (0, 1); turning points $(-1, 3)$ (maximum) $(1, -1)$ (minimum).
6. (3, 18) (maximum), $(-3, -18)$ (minimum); point of inflexion (0, 0).
7. $(1, -2)$, $(-1, 2)$.
8. Point of inflexion $(3, -\frac{2}{9})$; turning point (minimum) $(2, -\frac{1}{4})$.

## EXERCISE 3 N (p 143)

1 $v = 16, a = 22$ 2 (1) $v = -4, s = -\frac{16}{3}$, (2) $a = 8, s = 0$
3 (1) 23 f p s², (2) 12 f p s 4. (1) After 5 min , (2) at the start, $-75$ yd. per min , (3) $-74$ yd , $-72$ yd per min , 6 yd per min²
5. (1) After 4 sec , (2) after $1\frac{1}{3}$ sec , $6\frac{2}{3}$ f p s , (3) $5\frac{1}{3}$ ft , 6 f p s , 1 f p s²
6 20 7. $h$ 9 $\pm 22, -17\frac{1}{4}$ 10. (i) After 6 sec , $x = 108$, (ii) after 3 sec , 27 f p s , (iii) $-81$ f p s , (iv) $x = -100, v = -120$ f p s
11. The body starts with velocity $-3$ and moves in the direction of $s$ decreasing until $t = 1$, when the velocity is zero and the direction of motion is reversed When $t = \sqrt{3}$ it is back at the starting point with velocity 6 After that it moves in the direction of $s$ increasing
12 (a) 5 9 yd per sec², (b) 3 2 yd per sec² Maximum acceleration occurs at the start and is about 20 yd per sec² (i e about twice the acceleration of a falling stone) 7 0 yd per sec²

## EXERCISE 3 P (p 145)

1 70 ft 2. 12 f p s , 9 ft 3 $-43$ f p.s 4. 4 ft
5 (a) 116 ft , (b) 50 ft , $5\frac{5}{8}$ sec , $-90$ f p.s 6 30 ft
7 14 ft , 3 f p s , 0 8 10 sec , 52 8 ft 9. 4 sec , $-3\frac{1}{4}$ f p s
10 1,200 yd per min (= 41 m p h nearly), 1,200 yd
11. 800 yd per min. 12. 148 f p s , 336 ft
13 After 12 sec , $s = -432$ ft
14 It passes the starting point first after $1\frac{1}{2}$ sec with velocity $-7\frac{1}{2}$ and acceleration $-4$ and again after 4 sec with velocity 20 and acceleration 26 It changes direction first after $\frac{2}{3}$ sec at $s = 3\frac{13}{27}$ and again after 3 sec at $s = -9$
15 (a) $v = u+at, s = ut+\frac{1}{2}at^2$, (b) $v = u+\frac{1}{2}kt^2, s = ut+\frac{1}{6}kt^3$

## EXERCISE 3 Q (p 147)

1. $y = \frac{1}{12}x^4+ax+b, y = \frac{1}{12}x^4+x+1$ 2. $y = x^2+ax+b$
3 $y = 1/2x-1/6x^3+ax+b$ 4 $6y = x^3+3x^2+8x+12$
5. $3y = x-x^3$ 6 $2y = 2+x+2x^2-x^3$
7. $6y = x^3+6x, 6y-9x+2 = 0$ 8. $y = 2x^6+3x^2+3x+2$
9. $2y = 4x-x^3$ 10 $y = \frac{3}{4}x^4-24x+26$, minimum

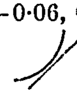
## EXERCISE 3 R (p 149)

1. $y = 2x^3-8x^2+4x+c$ 2. $y = c+\frac{1}{2}x^2+\frac{1}{3}x^3$
3 $y = \frac{1}{2}t^4-\frac{2}{3}t^3+t+c$ 4. $y = \frac{1}{3}x^3+1/3x^3+c$
5. $y = \frac{2}{3}x^3+ax+b$ 6 $h = \frac{1}{10}t^5+\frac{1}{3}t^3+at+b$ 7. $v = t+1/t+c$
8. $s = 1/t-4/t^2+at+b$ 9. $A = 6x^5-10x^3+4$
10. $v = \frac{1}{3}h^3-1/h+2$ 11. $s = \frac{1}{12}t^4-\frac{1}{3}t^3+4t$
12. $y = \frac{2}{3}x^3-\frac{5}{2}x^2+3x-\frac{5}{2}$ 13. $\theta = \frac{1}{2}t^2-\frac{1}{36}t^3-\frac{11}{18}t$
14 $y = \frac{3}{4}x^4$ 16. $3y = \frac{4}{3}x^3+c, 9y = 4x^3-5$ 17. $12y+x^4 = 36$
18. $xy = 2x^3+x+5$ 19. $y = \frac{16}{3}x^3+1/x-2, -4\frac{2}{3}$
21 $k = \frac{1}{8000}$, $B$ is 115 ft above sea level

## EXERCISE 3.S (p. 151)

1. $x\,dy/dx = 2y$. 2. $d^2y/dx^2 = 0$. 3. $x\,dy/dx+1 = y$.
4. $x\,d^2y/dx^2 = dy/dx$. 5. $x\,dy/dx+y = 0$.
6. $x^2\,d^2y/dx^2-2x\,dy/dx+2y = 0$. 7. $x\,dy/dx-2y = x^3$.
8. $x^2\,d^2y/dx^2+x\,dy/dx-y = 0$. 9. $x\,dy/dx = y$.
10. $dy/dx = 2$. 11. $dy/dx+1 = 0$. 12. $dy/dx = (y-2)/(x+1)$.

## MISCELLANEOUS EXERCISE 3.X (p. 151)

2. $9y+x = 20$. 3. $-\frac{1}{4}$, minimum. 4. $v = 6t^2-4$, $a = 12t$.
6. $14(1-6x^5)$, 5, maximum. 7. $a = 4$, $s = 2t^2+8t$.
8. $(-2, 16)$. 9. $y = 3+x^3-2x^5$. 10. 5″, 10″.
11. $dy = 3x^2\,dx$, 9·2. 12. 48. 13. $(-2, 2)$.
15. $v = 5-32t$, $a = -32$. 16. $v = u+10t$, $s = ut+5t^2$.
17. The gradient of the tangent is 4 and the gradient of the curve is decreasing. 
18. $6y = 1/x^2+3x+2$. 19. $x^2\,dy/dx-xy+2 = 0$.
20. $3y = 4+3x^2-x^3$. 21. 1,600 lb. wt.
22. $y+x = 1$, $y-x = 1$. 23. 2, maximum, $-2$, minimum.
24. 292, 288. 25. $4y+3x^4-4x^2-8x+5 = 0$.
27. Diameter 6 in., height 4 in. 28. $x\,dy/dx = 2y$.
29. $4y+x = 3$, $y-2x = 3$.
30. $y = \frac{1}{2}ax^2+x+c$. $x\,d^2y/dx^2-dy/dx+1 = 0$.
31. $x = 1$, $y = 2$. 32. $y = \frac{1}{20}x^2-1$. 33. 103.
34. $-1/a^2$; $x+a^2y = 2a$.
35. The gradient is $-1$ and is increasing. 
36. $v = 6t^2+4$, $s = 2t^3+4t$. 37. 6 f.p.s.², 128 ft. 38. $-6$.
39. 3 hr. 40. 3,300. 41. $2\pi x+6\pi/x^3$.
42. The sides parallel to $Ox$, $Oy$ are 4, 2. 43. $y = x-x^2+\frac{1}{3}x^3$.
44. (a) 1, (b) $\frac{1}{2}$, (c) $2x^2$. 45. $y = 2-x+\frac{1}{2}x^2-1/2x$.
46. $(\frac{3}{2}, -\frac{9}{2})$; $-4$, maximum; $-5$, minimum. 48. $-0{\cdot}06$, $0{\cdot}94$.
49. 1·2 f.p.s.², 19·2 ft. 50. Gradient 1 and increasing. 

## MISCELLANEOUS EXERCISE 3.Y (p. 154)

1. (a) $2x-x^7-\frac{5}{4}x^9$; (b) $2\pi(1+4/x^2)$; (c) $2(x^2-2a^2)/x^5$.
2. (a) $y = 9x+10x^3+5x^5+c$; (b) $y = x^3-2x-1/x+c$; (c) $y = -1/3x^3+1/5x^5+c$.
3. $y+3x = 4$, $3y-x = 7$, $(3, 3\frac{1}{3})$.
4. $4y+4x+9 = 0$, $x = \frac{5}{2}$, $4y+4x+5 = 0$. 5. $a^3y+2x = 3a$.
6. 0, maximum; $-8$, minimum; $(2, -4)$.
7. 1, maximum; 0, minimum. Symmetry about the $y$-axis.
8. $(2, 8)$ (maximum), $(-2, -8)$ (minimum), $(0, 0)$ point of inflexion.
9. 12 cm. per sec., 32 cm., 9 cm. per sec.
10. (i) $22\frac{1}{3}$ metres, (ii) $3\frac{1}{3}$ metres per min., (iii) 6 metres per min.²

**11** (i) 40 ft , (ii) 161° 50 , (iii) $\lambda = \frac{1}{115000}$, 170° 52 **12** 2
**13** So that they are equal 20 **14.** So that they are equal $\frac{1}{4}a^2$
**15** $r = (x^2+a^2)/2x$ $a$ The circular arc is a semicircle
**16** (a) $x = 4$ $y = 4$, (b) 125 cu in
**18** 120 lb per sq in (a) 1 344, (b) 0 384 ft per lb per sq in
**19** 355 metres **20** (a) 541,000, (b) 10 2 **21** 1 7 m p.h
**22** 16 1 m p h within about 0 06 **23** 510 ft
**24** 0, point of inflexion, $-\frac{27}{256}$, minimum
**25** $y = b+ax+x^2-\frac{4}{3}x^3$, $y = 2+x^2-\frac{4}{3}x^3$
**26** $a = -4$ $b = 5$, minimum **27** $y = 5+2x-\frac{1}{2}x^2$, maximum
**28** $y = 1+6x-3x^2-4x^3$, $-4$, minimum **29** $y+x = \frac{3}{10}$
**30** 4 yd per min $^2$, $6\frac{2}{3}$ yd , $3\frac{1}{2}$ yd per min , $\frac{256}{27}$ yd per min.
**31** 4 ft **32** 2,000 ft per min , 5 200 ft , 2 600 ft per min , (nearly 30 m p h ) **33** $y-8x = 12$, (1, 20)
**34** (i) $4x^3-4ax$, (ii) $-4\pi a/x^5-3\pi b/x^4$, (iii) $k(4t^3-2-2/t^3)$, $20/x^6-12/x^4$
**35** $x^2 d^2y/dx^2-2x\,dy/dx+2(y-1) = 0$
**36** $y = \frac{1}{2}x^2+1/2x^2+ax+b$ (a) $y = \frac{1}{2}x^2+1/2x^2+x-1$,
(b) $y = \frac{1}{2}x^2+1/2x^2-x$ (c) $y = \frac{1}{2}x^2+1/2x^2$ No
**37** 4 800 ft per min Yes
**38** (a) Increase of 37 lb wt , (b) decrease of 33 lb wt
**39** (a) $b = 12$ ft $d = 6$ ft (c) $b = 8$ ft , $d = 8$ ft
**40** $b = \sqrt{3} = 1\ 7$ ft , $d = \sqrt{6} = 2\ 4$ ft

## MISCELLANEOUS EXERCISE 3.Z (p 159)

**1** (i) $6x^2-14x$, (ii) $3ax^2+4b/x^5$, (iii) $10x^9-10/x^{11}$, (iv) $(5b-4at)/t^6$
**2** (i) $y = 2x^3-\frac{5}{4}x^2+4x+c$ (ii) $y = \frac{1}{3}x^3+a/x+c$,
(iii) $y = \frac{1}{4}x^4-ax^3+\frac{3}{2}a^2x^2-a^3x+c$
**3** $y-x+2 = 0$ $y+x = 0$ $y-x = 2$ $y+x = 0$ **4** $k = -2, 0$
**5** $y+x = 0$, $y-3x = 4$ **6** $(-8, 8)$, $2y-x = 24$, (12, 18)
**7** (i) 116 maximum $-100$, minimum (ii) $-54$
**8** $5\ 8\times 10^3$ ft per min
**11.** (i) 96 f p s , (ii) $s$ decreasing, (iii) after 4 sec , $-48$ f p.s $^2$, 256 ft
**12** $a = -8$, $b = 4$ $-12$
**13** $a = -1$ $b = -\frac{15}{4}$, $c = 3$, minimum, $x = -\frac{5}{6}$
**14** (i) Increase of 10 cwt , (ii) decrease of 4 cwt
**15** (a) 0, (b) $-0\ 034$, (c) temperature decreases, (d) $-34°$ C per km
**16** (a) The temperature of the body falls at a rate which is proportional to the excess of its temperature over that of its surroundings (b) $k = \frac{1}{35}$. (i) 43 sec (ii) $2\frac{1}{2}$ min
**17** (a) An increase of 1,500 lb wt , (b) a decrease of 320 lb wt
**20** $x = £50$ $n = 6$
**21.** The tangent and $OP$ are equally inclined to the axes $y = 4x$ (this is a normal at both its intersections with the curve)
**22** $y = \frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{2}x+c$, $y = \frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{2}x$ **23** $(1, -2)$
**25** $y = 0$ when $x = 0$ point of inflexion $y = 1$ when $x = 1$, maximum, $y = -27$ when $x = 3$, minimum

## CHAPTER IV

### EXERCISE 4.A (p. 166)

**1.** 39. **2.** 80. **3.** 6. **4.** $7\frac{1}{5}$. **5.** $6\frac{2}{3}$. **6.** 51. **7.** 18.
**8.** $17\frac{1}{3}$. **9.** 32·8. **10.** 5·2 acres. **11.** 60 sq. ft.
**12.** $42\frac{2}{3}$ sq. ft., $5\frac{1}{3}$ sq. ft. **13.** 80.

### EXERCISE 4.B (p. 167)

**1.** $10\frac{2}{3}$. **2.** 32. **3.** $x = 1, x = 3$; $x = 2$. **4.** 18.
**5.** $(-1, 1), (2, 1)$; $13\frac{1}{2}$. **6.** 128.

### EXERCISE 4.C (p. 169)

**1.** 2. **2.** (i) 4; (ii) 5; (iii) 1.004. **3.** $x = 3$.
**4.** (i) $x = 0$; (ii) $x = 2$.

## CHAPTER V

### EXERCISE 5.A (p. 175)

$34\frac{2}{3}$. (i) upper sum, 52; lower sum, 20. (ii) upper sum, 43; lower sum, 27.

### EXERCISE 5.B (p. 177)

**1.** The area under $y = x^2$ from $x = 1$ to $x = 2$. $2\frac{1}{3}$. **2.** 39.
**3.** 80. **4.** 130.

### EXERCISE 5.C (p. 178)

**1.** 56. **2.** 24. **3.** 45. **4.** $\frac{4}{5}$. **5.** 19·8. **6.** $\frac{3}{4}$.
**7.** 234. **8.** 5·5. **9.** 4·85. **10.** $30\frac{3}{4}$. **11.** $20\frac{5}{6}$. **12.** $4\frac{1}{8}$.
**13.** $17\frac{1}{2}$. **14.** (*a*) $21\frac{1}{2}$; (*b*) $57\frac{1}{6}$; (*c*) $78\frac{2}{3}$.

### EXERCISE 5.D (p. 181)

**1.** 2·4, its symmetry about the $y$-axis. **2.** $5\frac{1}{4}$. **3.** $67\frac{1}{2}$.
**4.** $3\frac{5}{12}$. **5.** 0·9, 0·333; $y = 1/x^4$ keeps closer. **6.** $2\frac{13}{36}$. **7.** 32.
**8.** $4\frac{1}{2}$. **9.** 3. **10.** $3\frac{2}{3}$. **11.** 12. **13.** 4:1. **14.** 82 tons.
**15.** $1\frac{1}{4}$, maximum; 1, minimum; 27.

### EXERCISE 5.E (p. 184)

**1.** 4. **2.** 36. **3.** $10\frac{2}{3}$. **4.** 16. **5.** $10\frac{2}{3}$. **6.** $6\frac{3}{4}$.
**7.** $4\frac{1}{2}$. **8.** 8. **9.** $6\frac{2}{3}$. **10.** $10\frac{2}{3}$ sq. in.

### EXERCISE 5.F (p. 187)

**1.** £74. 8*s*. **2.** 125 ft. **3.** $933\frac{1}{3}$ ft. **4.** 3,000 ft.
**5.** 96 sq. miles, 3·7 sq. miles. **6.** 11 ft. 4·8 in. **7.** 202 hr.

### EXERCISE 5.G (p. 190)

**1.** $124\pi/5$. **2.** $5\pi$. **3.** $21\pi$. **4.** $80\pi$. **5.** $250\pi/3$.
**6.** $y = rx/h$. **7.** (*a*) $8\frac{1}{2}$ in.; (*b*) $6\frac{2}{3}$ in., 11 in.; $72\frac{31}{48} \times \pi \doteqdot 228$ cu. in.
**8.** $136\pi \doteqdot 427$ cu. in. **9.** $17\pi/80$ cu. in. **10.** $\pi/30$.

## EXERCISE 5 H (p 191)

**1.** $15\pi/8$ **2.** $120\pi$ **3.** 4 in , $68\pi$ cu in **4.** (*a*) 3 in , (*b*) 1 in , $15\pi/2$ cu in , $5\pi$ cu in , $1\frac{1}{2}$ **5.** $\frac{2}{3}\pi a^3$, $\frac{4}{3}\pi a^3$ **6.** $2\ 32\pi$ cu in

## EXERCISE 5 J (p 193)

**1.** $16\pi/15$ **2.** $24\pi$ **3.** $\frac{2}{3}\pi$ **4.** $139\ 2\pi$. **5.** $16\pi/7$.
**6.** $128\pi/15$ **7.** $64\pi/5$ **8.** 2 7.

## EXERCISE 5 L (p 199)

**1.** 8 lb **2.** $12\frac{4}{5}$ oz **3.** 288 lb. **5** 6,675 lb
**6.** $41\ 6\pi$ lb $=$ 131 lb. **7.** 7,700 kg , 25 per cent

## EXERCISE 5 M (p 204)

**1.** 3 lb , 8 in **2.** 3 ft 6 in. **3.** 8 oz , $23\frac{1}{3}$ in , $11\frac{2}{3}$ in.

## EXERCISE 5 N (p 206)

**1.** $(1\frac{2}{5}, 0)$ **2.** $(1\frac{4}{7}, 0)$ **3.** $(3, 0)$ **4.** $(0, 2)$ **5.** $(3, 0)$
**7.** $(\frac{2}{3}h, 0)$ **9.** $(0, \frac{2}{3}k)$ **10.** $(\frac{11}{16}, 0)$

## EXERCISE 5 P (p 210)

**1.** $(1\ 5, 1\ 2)$ **2** $(\frac{4}{5}, \frac{5}{7})$ **3.** $(0, 1\ 6)$ **4.** $(\frac{3}{5}, \frac{2}{5})$
**5.** $(\frac{1}{2}, 1)$ **6.** (i) $(4, 4)$, (ii) $(2h/3, mh/3)$ **7.** $(1\frac{17}{28}, 3\frac{69}{70})$
**8.** $(1\ 4, 3\ 78)$ **9.** $2\frac{1}{6}$ **10.** $(\frac{2}{3}, 2\frac{1}{15})$

## EXERCISE 5 Q (p 211)

**1.** 30 cm per sec **2.** 840 yd per min **3.** 1,060 yd per min
**4.** 1 6 m p h , 3 m p h

## EXERCISE 5 R (p 213)

**1.** $9\frac{1}{3}$ **2.** 6 **3.** 16 **4.** 0 1 **5.** $3\frac{3}{4}$.
**7.** $30\frac{3}{5} = 30\ 6$ in , 30 5 in **8.** 6 ft , 8 ft **9.** $-\frac{5}{27}$, (1) 10 ft , (2) $8\frac{5}{9}$ ft **10.** 2 ft 8 in , 2 ft $4\frac{1}{2}$ in **11.** 73 75° F.

## EXERCISE 5 S (p 215)

**1.** 0 40(5) **2.** 5 3 **3.** 10 **4.** (*a*) 9, (*b*) $8\frac{3}{4}$

## EXERCISE 5 T (p 217)

**1.** About 240 cu in **2.** About 4 1 cu in **3.** About 80 cu ft
**4.** About 140,000 cu yd **5.** 360 lb **6.** $\frac{1}{3}\pi a^3$
**7.** 3 2 miles, 24 m p h **8.** 1 7 miles, 29 m p h

## EXERCISE 5 U (p 219)

**1.** 3 3 **2.** 8 8 **3.** 3 35 miles **4.** 92 or 93 kg
**5.** 7 5 sq ft

## MISCELLANEOUS EXERCISE 5 X (p 220)

**1.** 8 **2.** 78 **3.** 42 **4.** $127\pi/7$ **5** $350\pi = 1{,}100$ cu in
**6.** $11\frac{5}{6}$. **7.** $5\frac{5}{24}$ **9** $4y = x+2$, $7\pi/6$ ($= 3\ 7$) cu ft
**10.** $(\frac{7}{8}, 0)$ **11.** $\frac{1}{6}$ **12.** $9\frac{1}{2}$ **13.** $60\pi$ **14.** $\frac{4}{3}$
**15.** $y = x^3+x+2$ **16.** $(\frac{9}{16}, \frac{7}{10})$ **17.** 4 **18.** $2\sqrt{2}$

**19.** 159. **20.** £1. 5*s*. **21.** $56\frac{1}{4}$. **22.** $1\frac{1}{15}$. **23.** $112\pi/15$.
**25.** $16\frac{2}{3}$. **26.** (*a*) 60 m.p.h.; (*b*) 40 m.p.h. **27.** (*a*) $8\pi$; (*b*) $4\pi/5$.
**28.** (2, 0). **29.** $AB \equiv y = 12+x-x^2$, $BC \equiv y = 12-4x-x^2$. $35\frac{5}{6}$ sq. chains. **30.** 0·55. **31.** $5\pi/3$. **32.** (0, 2·4).
**33.** $13\frac{1}{2}$. **34.** 700 cm. **35.** $c = \frac{3}{2}$. **36.** $\frac{1}{2}\pi$. **37.** $64\pi/15$.
**38.** (*a*) $233\frac{1}{3}$; (*b*) 235. **39.** 108 lb.; 1 ft. 7 in.

MISCELLANEOUS EXERCISE 5.Y (p. 223)

**1.** 4. **2.** 4. **3.** $85\frac{1}{3}$. **6.** $11{\cdot}2\pi \doteqdot 35{\cdot}2$.
**8.** $112\pi/3 \doteqdot 117$ cu. in. **9.** (i) 6 lb.; (ii) $3\frac{3}{8}$ ft.
**10.** $3000\pi$ ($\doteqdot 9420$) cu. ft. **11.** 76,800 lb. wt. ($\doteqdot 34{\cdot}3$ tons wt.).
**12.** $29\pi/5$. **13.** (*a*) $2\pi$; (*b*) $101\pi/5$. **16.** $29\frac{1}{4}$. **18.** $\frac{3}{4}h$.
**19.** 16 ft.; maximum; $\pm 12$ f.p.s.$^2$ **21.** 0·17. **22.** $2\frac{1}{4}$ cu. ft.
**23.** 0·33. **24.** $2\frac{2}{3}$. **25.** $13\frac{1}{3}$; (0, 2). **26.** $(1\frac{1}{3}, 3\frac{3}{5})$.
**27.** 8·4 lb. wt. **28.** (i) $2h:5$; (ii) $h = 2\frac{1}{2}$. **29.** $52\frac{1}{6}$ ft.
**30.** ($2\frac{2}{3}\times10^4$ sq. yd.) $\doteqdot$ 6 acres. **31.** $6\frac{2}{3}$ in.; 4 in.; about 320 cu. in.
**32.** 32. **33.** $5\frac{1}{3}$, $2\pi$. **34.** 6; $(1\frac{3}{5}, 2\frac{5}{14})$, $198\pi/7$. **35.** $7\pi/24$.

MISCELLANEOUS EXERCISE 5.Z (p. 226)

**1.** 32. **4.** $a^3/6k^2$, a decrease of about 0·11. **5.** 1:4.
**6.** (*a*) $153\pi/16$. **7.** $y+x = 3$, $11\pi/15$. **8.** 2:7. **9.** 34, 6·8.
**10.** $\frac{1}{2}\pi(3h^2+2h)$, an increase of about 0·1 in.
**11.** $\sqrt{(3h^2+4)}$, $3y^2 = x^2-4$. **12.** 3,600 lb. wt. **13.** $(3\frac{1}{3}, 1)$.
**14.1** About 198 cu. ft. **14.3** 64·4 cu. ft. **14.5** (*a*) 0·70; (*b*) 1·41. **14.6.** 8·58 ft. **15.** (*a*) $4y-x = 12$; $392\pi/3$ cu. in.; (*c*) $38\pi$ ($\doteqdot 119$) cu. ft. **16.** 88 lb. **18.** $56\pi$. **19.** $3\frac{3}{8}$. **20.** $a/2$.
**21.** (*a*) $\pi a^3/5$; (*b*) $\pi a^3/2$; (*c*) $8\pi a^3/15$. **22.** $2\frac{1}{4}$. **23.** (*a*) $\pi/105$; (*b*) $59\pi/70$.; $(\frac{5}{8}, 0)$. **24.** $v = u+kt^3/3$, $s = ut+kt^4/12$.
**25.** 68·4° F.; 63° F.; 54° F.; 71° F. at 4 p.m.; the graph of the formula shows a point of inflexion with zero gradient when $t = 0$.
**26.** $(3a/8, 2a/5)$. **27.** $a = 1$, $y = 6+4x-2x^2$; 8.
**28.** $a = 0$, $b = 2$, $c = -1$; (0, 0); $16\pi/15$.
**29.** (1) 8; (2) $\frac{1}{4}\pi$ cu. ft.; (3) $Q = k\pi r^4/2$. **30.** About $45\frac{1}{2}$ cu. in.
**31.** $\pi a^2/3$.

# INDEX

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